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A FRAMEWORK FOR THE EXACT EVALUATION OF EXPECTED CYCLE TIMES IN AUTOMATED STORAGE SYSTEMS WITH FULL-TURNOVER ITEM ALLOCATION AND RANDOM SERVICE REQUESTS

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Abstract—We present a new framework for obtaining analytic expressions of the expected throughput rate in Automated Storage and Retrieval Systems (AS/RSs). The models developed consider generalized full-turnover item allocation policies and random storage and retrieval requests. Both single command and dual command operations are considered. A general expression for the expected cycle time in a single command class-based system is also developed. It is shown that some well known results are derived as special cases of our expressions. These results provide for rapid performance evaluation of various operational policies in automated warehouses and in certain mass-storage information devices currently used by various computer systems. The potential for significant reductions in the expected cycle time is demonstrated in the case of full-turnover item allocation.

1. INTRODUCTION

Automated Storage and Retrieval Systems(AS/RSs) are in frequent use in numerous manufacturing and distribution centers around the world. These systems perform material storage and retrieval functions in a fully or semi-automated manner under the control of real-time computer systems. A typical system is composed of multiple parallel aisles of storage racks, a storage/retrieval crane for each aisle, and an input/output (I/O) pickup and deposit station. The crane has horizontal and vertical drives which operate simultaneously in order to reduce the travel time. As a result the distance between any two points is measured by the Chebyshev (or the l_∞-norm) metric. In a dual-command operational mode each crane cycle begins with the crane at the I/O point; it picks up a load, travels to the designated storage location, deposits the load, travels empty (interleaves) to the other designated retrieval location, retrieves another load, and then travels back to the I/O point and deposits the load there. In a single command mode there is no interleaving and the crane returns to the I/O point following each storage or retrieval task. Several warehousing and manufacturing applications of AS/RSs are discussed by Seidmann [1] and Sule [2]. Information storage and retrieval applications in two-dimensional mass storage systems are detailed by Wong [3] and others.

There are many benefits to AS/RSs, such as reduced labor costs, high floor and cube space utilization, improved material flow and inventory controls. Unlike traditional warehouses, where the data record of location and inventory levels are collected manually, in AS/RSs such controls are maintained by the supervisory computer system. These controls account for significant savings in the inventory holding costs. In addition, using high-density high-rise racks reduce travel times and floor space. These AS/RS systems do, however, require a high initial investment, and a through analysis to determine their economic viability. Realizing the benefits of these service systems depends on specifying the appropriate AS/RS configuration and on the development of effective management policies for operating these systems. Analytical models, in particular closed-form expressions for determining the system performance, are extremely useful for the rapid exploration of various operational policies and of potential economic savings.

This paper presents exact closed-form expressions for computing the throughput rate of an AS/RS with full-turnover item allocation to storage and with random storage and retrieval requests. It considers both single command and dual command crane movements. Full-turnover allocations are commonly used in mini-load systems where storage containers with frequently kitted components are given preferential storage assignments. Using simple pairwise switching arguments Hausman *et al.* [4] show that the expected one-way travel time is minimized under this policy. The studies by Hausman *et al.* [4] and Graves *et al.* [5] present closed form expressions for both single and dual command cycle times under several storage allocation policies. A power function is used by these authors to describe the cumulative percentage demand for various items. This function limits the applicability of their models to square-in-time (SIT) shapes as discussed later. Bozer and White [6] deal with non-square-in-time (NSIT) shape warehouse which includes square-in-time (SIT) as a special case. They use an order statistic method to derive the expected interleaving time assuming uniform (or random) storage allocation policy. Foley and Frazelle [7] investigate the cycle times properties of an SIT miniload system with head of the aisle pick delays and uniform storage locations. We propose a more general functional form to represent the relative frequency of storage/retrieval requests under full-turnover storage policy. Our framework is detailed for a specific turnover function, but it can accommodate many other two dimensional density structures.

Section 2 presents the model parameters and the probability density function (pdf) which describes the relative demand frequencies of the items in storage. The expected single command time is derived in Section 3 and Section 4 analyzes the performance of the class-based policy. Sections 5 and 6 derive the expected dual command cycle time. Section 7 presents a comparative evaluation of the random storage and the full-turnover based policies.

2. THE MODEL

A storage rack is said to be SIT if the time required to traverse the entire length of the rack (front to back) is equal to the time required to traverse the entire height of the rack (bottom to top). A rack may be SIT but not square physically as its motors may have different speeds. Following on Hausman *et al.* [4] we use a unit of travel time measurement to yield a storage space with unit sides: the origin (0, 0) is assumed to be the I/O point; the X-axis and Y-axis represent the horizontal and vertical travel time, respectively. Since the storage cell is small enough compared with the vertical cross-section of a warehouse any point $(x, y) \in \{(u, v) | 0 < u \leqslant 1, 0 < v \leqslant 1\}$ is assumed to represent the location of a storage cell. We define in the above unit square a random vector $\mathbf{X} = (X, Y)$ and its pdf

$$f_{X,Y}(x, y) = \begin{cases} \alpha\, \mathrm{e}^{-\lambda x} & \text{if} \quad 1 \geqslant x \geqslant y > 0 \\ \alpha\, \mathrm{e}^{-\lambda y} & \text{if} \quad 0 < x \leqslant y \leqslant 1 \end{cases} \tag{1}$$

where

$$\alpha = \begin{cases} \dfrac{\lambda^2 \mathrm{e}^{\lambda}}{2(\mathrm{e}^{\lambda} - \lambda - 1)} & \text{if} \quad \lambda > 0 \\ 1 & \text{if} \quad \lambda = 0 \end{cases} \tag{2}$$

and the cumulative probability is

$$F_{X,Y}(c, c) = \int_0^c \int_0^c f_{X,Y}(x, y)\, \mathrm{d}x\, \mathrm{d}y$$

$$= (\mathrm{e}^{\lambda c} - \lambda c - 1)\left\{\frac{\mathrm{e}^{\lambda(1-c)}}{\mathrm{e}^{\lambda} - \lambda - 1}\right\}. \tag{3}$$

This probability function is assumed to be known and stationary. The probability mass of the vector $\mathbf{X}$ has its highest value at $(0^+, 0^+)$ and reduces as $X \to 1$ and $Y \to 1$. It is used in representing the traffic of items stored in the vertical unit square according to their relative demand, or turnover frequency. Under this storage policy the distance (in time) of each item from the I/O point is proportional to its relative demand. Hence, the fastest moving items are closest to the I/O point. The crane serving this area can simultaneously move both vertically and horizontally and at different velocities.

When the items in storage are ranked according to their relative contribution to the total crane activities we get the well known pareto (or "ABC") curves. Figure 1 depicts these curves for several

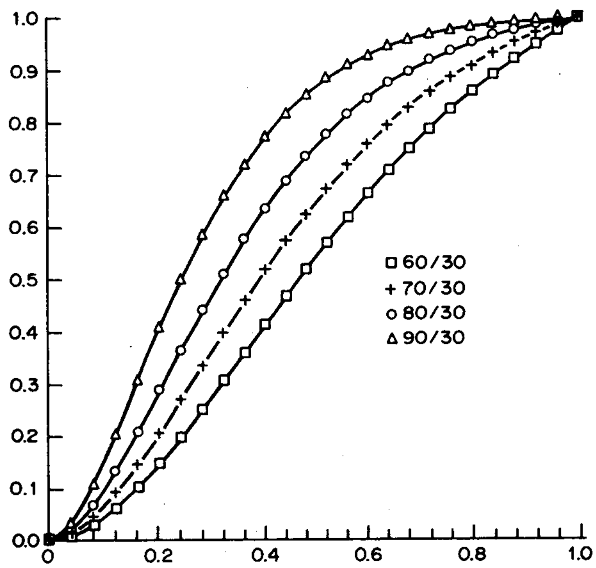

Fig. 1. Cumulative storage/retrieval request rates of some turnover functions.

cases. The pdf defined in equation (1) above can be fitted to empirical data by adjusting the shape parameter λ. For example, in the case of an 80/30 allocation one expects that 30% of the items stored account for 80% of the total storage/retrieval activities. Getting the right value of λ simply solve equation (3) for λ using $F_{X,Y}(\sqrt{0.3}) = 0.8$. Figure 2 displays the pdf of the 80/30 distribution ($\lambda = 5.136$). Note that when $\lambda \rightarrow 0^+$ the above two dimensional pdf reduces to the uniform distribution.

Now we define two new random variables, T_1 and T_2, which represent the travel (interleaving) time from the I/O point to a point $\mathbf{X} = (X, Y)$ and the travel time between two random points $\mathbf{X}_1 = (X_1, Y_1)$ and $\mathbf{X}_2 = (X_2, Y_2)$ in the unit square, respectively:

$$T_1 = T_1(\mathbf{X}) = \max\{X, Y\} \tag{4}$$

and

$$T_2 = T_2(\mathbf{X}_1, \mathbf{X}_2) = \max\{|X_1 - X_2|, |Y_1 - Y_2|\}. \tag{5}$$

Then we immediately obtain the following relations:

$$E[SC] = 2E[T_1] \tag{6}$$

$$E[DC] = 2E[T_1] + E[T_2] \tag{7}$$

where $E[SC]$ and $E[DC]$ represent the expected crane travel time of one round trip of the single command and the dual command, respectively. Here we assume that two random points along the dual command cycle are independent.

3. EXPECTED TRAVEL TIME OF SINGLE COMMAND

Computing $E[T_1]$ we start by partitioning the unit square into regions $A = \{(x, y) | 0 < x \leqslant y \leqslant 1\}$ and $B = \{(x, y) | 0 < y \leqslant x \leqslant 1\}$ as in Fig. 3. Then we have

$$T_1 = \begin{cases} Y & \text{if } \mathbf{X} \in A \\ X & \text{if } \mathbf{X} \in B \end{cases} \tag{8}$$

and

$$f_{\mathbf{X}}(x, y) = \begin{cases} \alpha\, \mathrm{e}^{-\lambda x} & \text{if } \mathbf{X} \in B \\ \alpha\, \mathrm{e}^{-\lambda y} & \text{if } \mathbf{X} \in A. \end{cases} \tag{9}$$

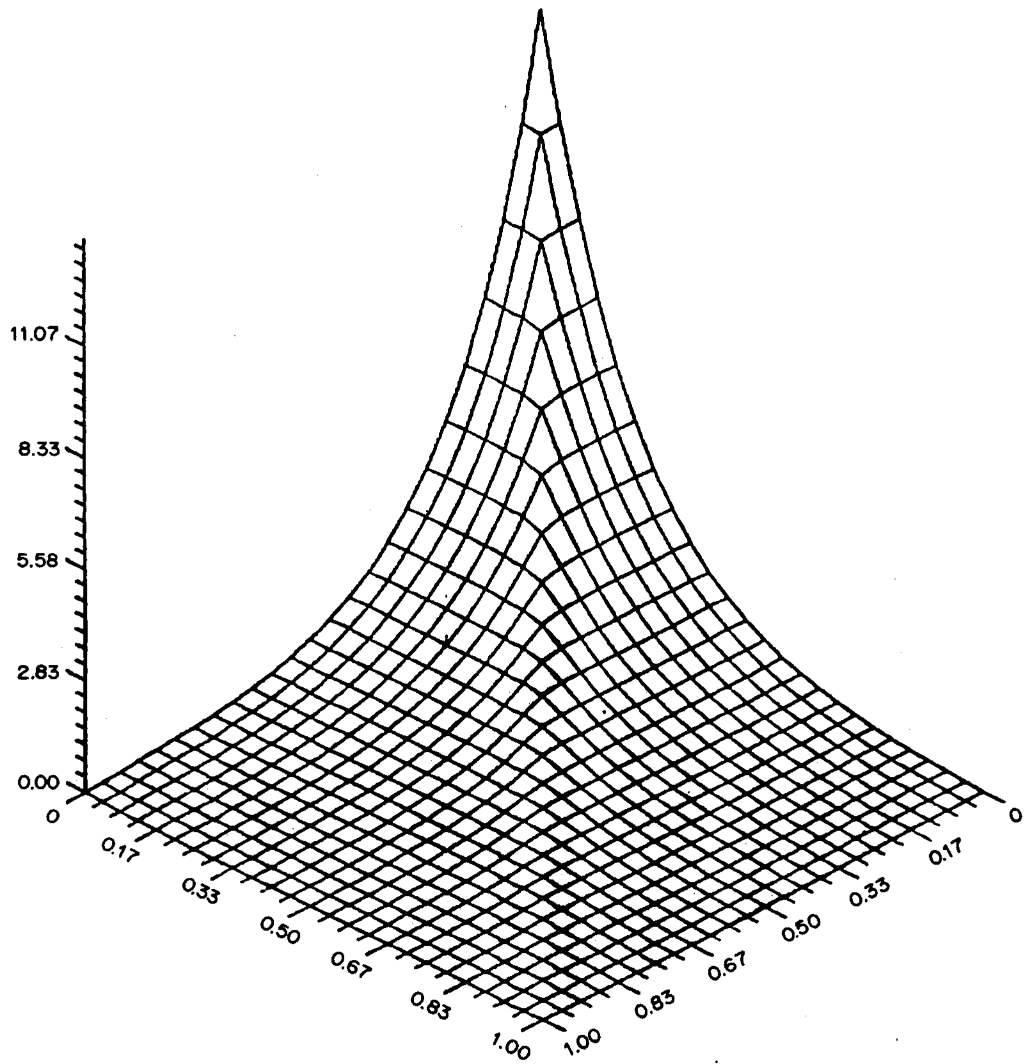

Fig. 2. The probability density of 80/30.

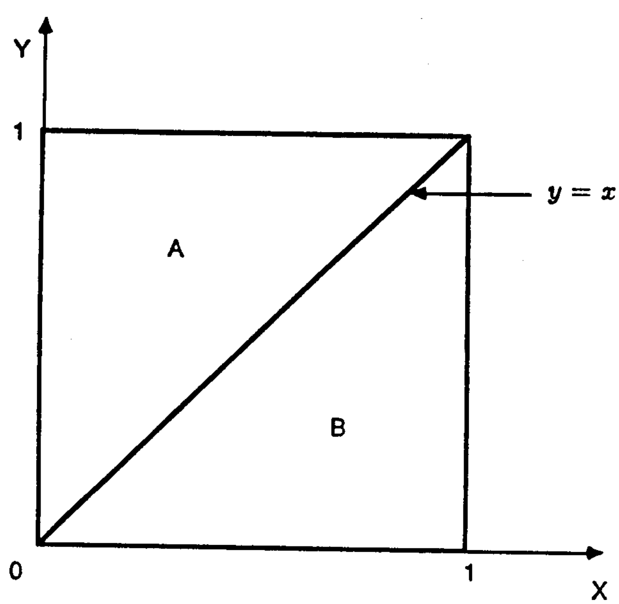

Fig. 3. Partition of the unit square.

Thereby, denoting t a realization of T_1, we get

$$E[T_1] = \int_{\mathbf{X}\in A} t(\mathbf{x}) f_{\mathbf{X}\in A}(\mathbf{x}) + \int_{\mathbf{X}\in B} t(\mathbf{x}) f_{\mathbf{X}\in B}(\mathbf{x})$$

$$= \int_0^1 \int_x^1 y\alpha\, \mathrm{e}^{-\lambda y}\, \mathrm{d}y\, \mathrm{d}x + \int_0^1 \int_0^x x\alpha\, \mathrm{e}^{-\lambda x}\, \mathrm{d}y\, \mathrm{d}x$$

$$= \frac{2\,\mathrm{e}^{\lambda} - \lambda^2 - 2\lambda - 2}{\lambda(\mathrm{e}^{\lambda} - \lambda - 1)}. \tag{10}$$

Hence we obtain from (6) that

$$E[SC] = \frac{4\,\mathrm{e}^{\lambda} - 2\lambda^2 - 4\lambda - 4}{\lambda(\mathrm{e}^{\lambda} - \lambda - 1)}. \tag{11}$$

Note that, in the case of random storage (i.e. uniform item distribution), $\lim_{\lambda \to 0+} E[T_1] = 2/3$. This conforms with the earlier result of Hausman *et al.* [4].

4. EXPECTED TRAVEL TIME OF SINGLE COMMAND UNDER CLASS-BASED STORAGE ASSIGNMENT POLICY

Class-based storage allocation policy is commonly practiced in numerous warehouses when the actual turnover rate is not uniform. In this policy items are classified based on their storage/retrieval request frequency and are then stored in corresponding segments of the warehouse as shown in Fig. 4. The fastest moving items are classified as class 1 items and they are assigned to the segment nearest (in time) to the I/O point, the next subset of items is assigned to the following segment and so on. It is assumed that the items within each class are stored at an empty rack which is closest (in time) to the I/O point. This results in a uniform item storage distribution within each class.

The distribution function $F_{X,Y}(c, c)$ defined by equation (3) is used to approximate the actual distribution of storage/retrieval frequency data. Let p_i be the management decision on the portion of the total storage/retrieval request frequency allocated to class i items. Let t_i be the border point on the X-axis (and the Y-axis) which separates class i and class $i+1$ segments, for $i = 1, \ldots, n$, $t_0 = 0$, and $t_n = 1$. Then we have

$$(t_i) = F_{X,Y}^{-1}\left(\sum_{j=1}^{i} p_j\right), \quad i = 1, \ldots, n. \tag{12}$$

Furthermore, since $f_{X,Y}(x, y)$ is a monotone decreasing function, we have

$$t_i^2 \leqslant \sum_{j=1}^{i} p_j.$$

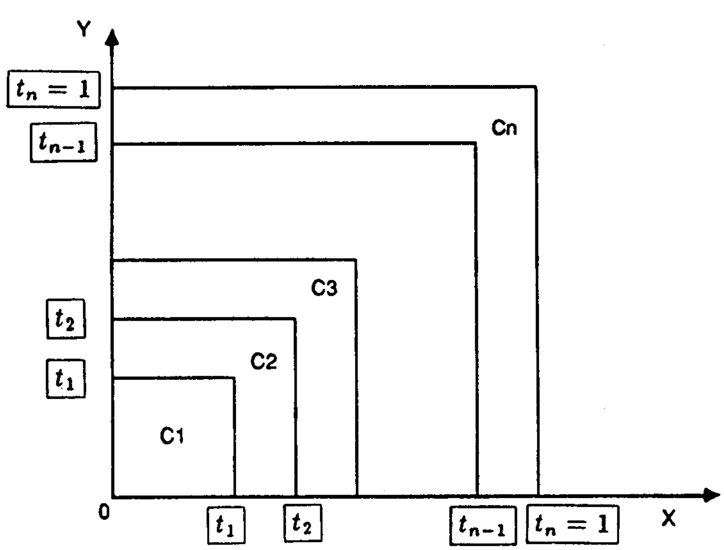

Fig. 4. N-class partition of the warehouse under Class-based item allocation policy.

Let $h_{\mathbf{X}|\mathbf{X}\in C_i}(\mathbf{x})$ be the pdf of a storage coordinates vector $\mathbf{X}$ given that it is in class i. The assumption of uniform distribution within each class gives us:

$$h_{\mathbf{X}|\mathbf{X}\in C_i}(\mathbf{x}) = \frac{1}{t_i^2 - t_{i-1}^2}, \quad i = 1, \ldots, n. \tag{13}$$

For each class i, the conditional expected travel time of single command is given by

$$\begin{aligned} E[SC]\mathbf{X}\in C_i] &= 2\int_{\mathbf{X}\in C_i} t(\mathbf{x}) h_{\mathbf{X}|\mathbf{X}\in C_i}(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &= 2\left\{\int_{t_{i-1}}^{t_i}\int_0^x x\frac{1}{t_i^2 - t_{i-1}^2}\,\mathrm{d}y\,\mathrm{d}x + \int_{t_{i-1}}^{t_i}\int_0^y y\frac{1}{t_i^2 - t_{i-1}^2}\,\mathrm{d}x\,\mathrm{d}y\right\} \\ &= \frac{4}{3}\left\{\frac{t_i^3 - t_{i-1}^3}{t_i^2 - t_{i-1}^2}\right\}. \end{aligned} \tag{14}$$

Hence our general expression for the expected travel time of a single command under n-class-based storage assignment policy is given by:

$$\begin{aligned} E_{n\text{-class}}[SC] &= \sum_{i=1}^{n} E[SC\,|\,\mathbf{X}\in C_i]Pr[\mathbf{X}\in C_i] \\ &= \frac{4}{3}\sum_{i=1}^{n} p_i\left\{\frac{t_i^3 - t_{i-1}^3}{t_i^2 - t_{i-1}^2}\right\}. \end{aligned} \tag{15}$$

Expression (15) extends the earlier results presented by Hausman *et al.* [4] for the cases of two and three classes.

For example, in the special case of a two-class system and $\mathbf{X}\in C_2$, we get

$$E[SC\,|\,\mathbf{X}\in C_2] = \frac{4}{3}\left\{\frac{1 - t_1^3}{1 - t_1^2}\right\}, \tag{16}$$

which conforms with the earlier result of Hausman *et al.* [4].

The value of $E_{n\text{-class}}[SC]$ depends on the shape of the actual turnover rate distribution and on the management decision regarding the number of classes (i.e., n) and the relative fraction of the cumulative turnover rate assigned to each class (i.e., $p_1, \ldots, p_n$).

5. EXPECTED INTERLEAVING TRAVEL TIME OF DUAL COMMAND

Since we have already computed $E[T_1]$, it suffices to compute $E[T_2]$ in order to compute $E[DC]$ by equation (7). Based on the partitioning in Fig. 1, the whole state space of $(\mathbf{X}_1, \mathbf{X}_2)$, where $\mathbf{X}_1$ and $\mathbf{X}_2$ are two points on the cycle of a dual command, can be partioned into four substates:

$$S_1 = \{\mathbf{X}_1\in A, \mathbf{X}_2\in A\} \tag{17}$$

$$S_2 = \{\mathbf{X}_1\in A, \mathbf{X}_2\in B\} \tag{18}$$

$$S_3 = \{\mathbf{X}_1\in B, \mathbf{X}_2\in B\} \tag{19}$$

$$S_4 = \{\mathbf{X}_1\in B, \mathbf{X}_2\in A\}. \tag{20}$$

Let t be a realization of T_2 and define

$$E_i[T_2] \overset{\text{def}}{=} \int_{(\mathbf{X}_1,\mathbf{X}_2)\in S_i} t f_{\mathbf{X}_1,\mathbf{X}_2}(\mathbf{x}_1, \mathbf{x}_2)\,\mathrm{d}\mathbf{x}_1\,\mathrm{d}\mathbf{x}_2. \tag{21}$$

From the above definition we can express $E[T_2]$ as the following sum:

$$E[T_2] = \sum_{i=1}^{4} E_i[T_2]. \tag{22}$$

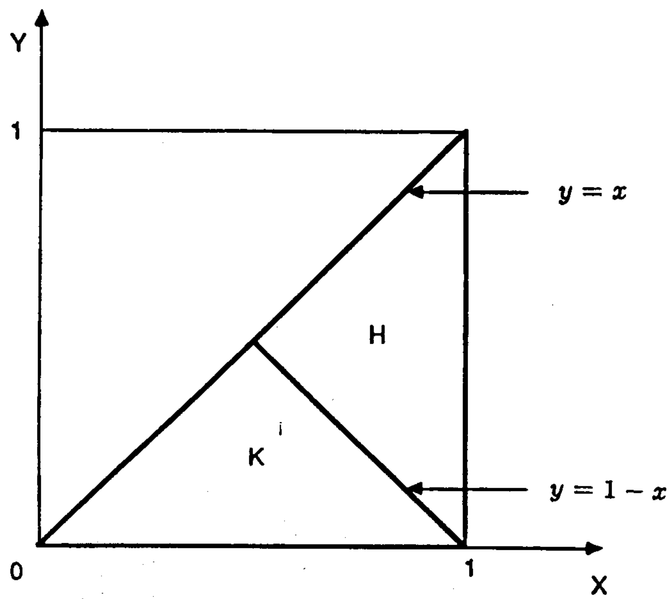

Fig. 5(a). Partition of region B.

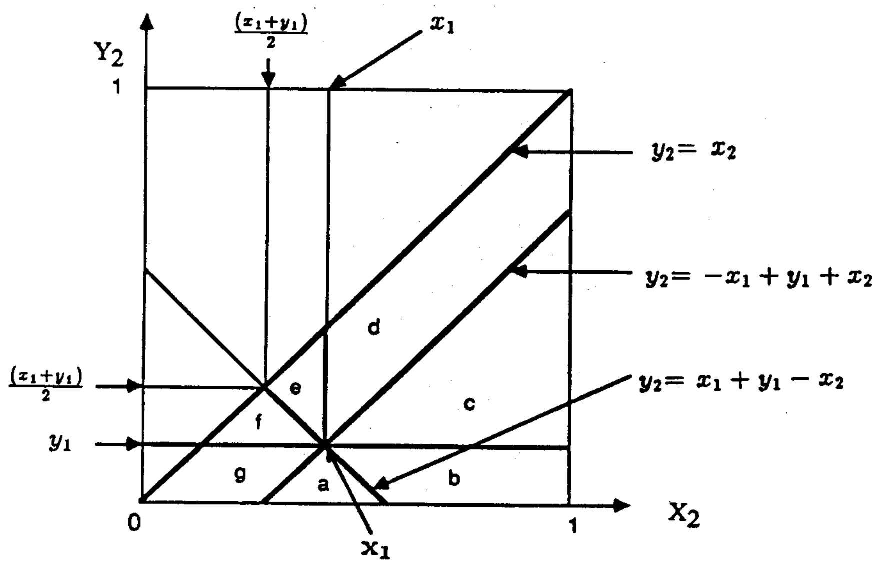

Fig. 5(b). Detailed partition of region B.

Using the independence of the random vectors and the symmetry of $f_{X,Y}(x, y)$ about $x = y$, it is obvious that $E_1[T_2] = E_3[T_2]$ and $E_2[T_2] = E_4[T_2]$. Hence it suffices to compute, say only $E_3[T_2]$ and $E_4[T_2]$, and then we get the desired expectation by the relation

$$E[T_2] = 2(E_3[T_2] + E_4[T_2]). \tag{23}$$

The following subsections compute $E_3[T_2]$ and $E_4[T_2]$.

5.1. Computation of $E_3[T_2]$

We partition the region B in Fig. 3 in two ways. We first partition B into two regions K and H [Fig. 5(a)] and then we partition B into seven other regions a, b, c, d, e, f, g [Fig. 5(b)].

Having $K \cup H = B$ leads to the following relations:

$$E_3[T_2] = \int_{\mathbf{x}_1 \in K} \int_{\mathbf{x}_2 \in B} t(\mathbf{x}_1, \mathbf{x}_2) f_{\mathbf{x}_2 | \mathbf{x}_1 \in K}(\mathbf{x}_2)\, d\mathbf{x}_2 f_{\mathbf{x}_1}(\mathbf{x}_1)\, d\mathbf{x}_1 + \int_{\mathbf{x}_1 \in H} \int_{\mathbf{x}_2 \in B} t(\mathbf{x}_1, \mathbf{x}_2) f_{\mathbf{x}_2 | \mathbf{x}_1 \in H}(\mathbf{x}_2)\, d\mathbf{x}_2 f_{\mathbf{x}_1}(\mathbf{x}_1)\, d\mathbf{x}_1. \tag{24}$$

Note that the first term in equation (24) can be expressed as a function of the seven subregions of B:

$$\int_{\mathbf{X}_1\in K}\int_{\mathbf{X}_2\in B} t(\mathbf{x}_1,\mathbf{x}_2)f_{\mathbf{X}_2|\mathbf{X}_1\in K}(\mathbf{x}_2)\,d\mathbf{x}_2 f_{\mathbf{X}_2}(\mathbf{x}_1)\,d\mathbf{x}_1$$

$$=\int_{\mathbf{X}_1\in K}\left(\sum_{S=a}^{g}\int_{\mathbf{X}_2\in S} t(\mathbf{x}_1,\mathbf{x}_2)f_{\mathbf{X}_2}(\mathbf{x}_2)\,d\mathbf{x}_2\right)f_{\mathbf{X}_1}(\mathbf{x}_1)\,d\mathbf{x}_1. \quad (25)$$

Note that the partitioning of B into seven subregions makes it easy to express t uniformly as function of x and y in each of them. For example, $t=(y_1-y_2)$ for $\mathbf{X}_1\in K$ and $\mathbf{X}_2\in a$. Next, we develop expressions for

$$\int_{\mathbf{X}_2\in S|\mathbf{X}_1\in K} t(\mathbf{x}_1,\mathbf{x}_2)f_{\mathbf{X}_2}(\mathbf{x}_2)\,d\mathbf{x}_2$$

for $S=a$, b, c, d, e, f, and g. Doing so we have obtained the following seven equations:

$$\int_{\mathbf{X}_2\in a|\mathbf{X}_1\in K} t(\mathbf{x}_1,\mathbf{x}_2)f_{\mathbf{X}_2}(\mathbf{x}_2)\,d\mathbf{x}_2=\int_0^{y_1}\int_{x_1-y_1+y_2}^{x_1+y_1-y_2}(y_1-y_2)\alpha\,e^{-\lambda x_2}\,dx_2\,dy_2 \quad (26)$$

$$\int_{\mathbf{X}_2\in b|\mathbf{X}_1\in K} t(\mathbf{x}_1,\mathbf{x}_2)f_{\mathbf{X}_2}(\mathbf{x}_2)\,d\mathbf{x}_2=\int_0^{y_1}\int_{x_1+y_1-y_2}^{1}(x_2-x_1)\alpha\,e^{-\lambda x_2}\,dx_2\,dy_2 \quad (27)$$

$$\int_{\mathbf{X}_2\in c|\mathbf{X}_1\in K} t(\mathbf{x}_1,\mathbf{x}_2)f_{\mathbf{X}_2}(\mathbf{x}_2)\,d\mathbf{x}_2=\int_{x_1}^{1}\int_{y_1}^{-x_1+y_1+x_2}(x_2-x_1)\alpha\,e^{-\lambda x_2}\,dy_2\,dx_2 \quad (28)$$

$$\int_{\mathbf{X}_2\in d|\mathbf{X}_1\in K} t(\mathbf{x}_1,\mathbf{x}_2)f_{\mathbf{X}_2}(\mathbf{x}_2)\,d\mathbf{x}_2=\int_{x_1}^{1}\int_{-x_1+y_1+x_2}^{x_2}(y_2-y_1)\alpha\,e^{-\lambda x_2}\,dy_2\,dx_2 \quad (29)$$

$$\int_{\mathbf{X}_2\in e|\mathbf{X}_1\in K} t(\mathbf{x}_1,\mathbf{x}_2)f_{\mathbf{X}_2}(\mathbf{x}_2)\,d\mathbf{x}_2=\int_{(x_1+y_1)/2}^{x_1}\int_{x_1+y_1-x_2}^{x_2}(y_2-y_1)\alpha\,e^{-\lambda x_2}\,dy_2\,dx_2 \quad (30)$$

$$\int_{\mathbf{X}_2\in f|\mathbf{X}_1\in K} t(\mathbf{x}_1,\mathbf{x}_2)f_{\mathbf{X}_2}(\mathbf{x}_2)\,d\mathbf{x}_2=\int_{y_1}^{(x_1+y_1)/2}\int_{y_2}^{x_1+y_1-y_2}(x_1-x_2)\alpha\,e^{-\lambda x_2}\,dx_2\,dy_2 \quad (31)$$

$$\int_{\mathbf{X}_2\in g|\mathbf{X}_1\in K} t(\mathbf{x}_1,\mathbf{x}_2)f_{\mathbf{X}_2}(\mathbf{x}_2)\,d\mathbf{x}_2=\int_0^{y_1}\int_{y_2}^{x_1-y_1+y_2}(x_1-x_2)\alpha\,e^{-\lambda x_2}\,dx_2\,dy_2. \quad (32)$$

Integration of the individual expression of the above set of equations can be done directly. Summing up results of the actual integrations of equations (26–32) we obtain

$$\int_{\mathbf{X}_2\in B} t(\mathbf{x}_1,\mathbf{x}_2)f_{\mathbf{X}_2|\mathbf{X}_1\in K}(\mathbf{x}_2)\,d\mathbf{x}_2=\alpha\left\{\frac{1}{\lambda^3}\left(e^{-\lambda(x_1-y_1)}-e^{-\lambda(x_1+y_1)}+4e^{-\lambda/2(x_1+y_1)}\right)-\frac{e^{-\lambda}}{2\lambda}(x_1-y_1)^2\right.$$

$$\left.+\left(\left(\frac{1}{\lambda}+\frac{1}{\lambda^2}\right)e^{-\lambda}+\frac{1}{\lambda^2}\right)x_1-\left(\frac{1}{\lambda}+\frac{2}{\lambda^2}+\frac{2}{\lambda^3}\right)e^{-\lambda}+\frac{2}{\lambda^3}\right\}. \quad (33)$$

Integrating equation (33) over $\mathbf{X}_1\in K$ yields for the first term in equation (24) in which $\mathbf{X}_1\in K$ and $\mathbf{X}_2\in B$

$$\int_{X_1 \in K} \int_{X_2 \in B} t(\mathbf{x}_1, \mathbf{x}_2) f_{X_2|X_1 \in K}(\mathbf{x}_2)\, d\mathbf{x}_2 f_{X_1}(\mathbf{x}_1)\, d\mathbf{x}_1$$

$$= \int_0^{1/2} \int_{y_1}^{1-y_1} \left(\sum_{S=a}^{g} \int_{X_2 \in S} t(\mathbf{x}_1, \mathbf{x}_2) f_{X_2}(\mathbf{x}_2)\, d\mathbf{x}_2 \right) f_{X_1, Y_1}(x_1, y_1)\, dx_1\, dy_1$$

$$= \alpha^2 \left\{ -\left(\frac{1}{2\lambda^3} + \frac{2}{\lambda^4} + \frac{22}{3\lambda^5} \right) e^{-2\lambda} + \left(\frac{1}{\lambda^3} - \frac{1}{\lambda^4} + \frac{34}{3\lambda^5} \right) e^{-3/2\lambda} \right.$$

$$\left. - \left(\frac{1}{\lambda^3} - \frac{1}{\lambda^4} + \frac{5}{\lambda^5} \right) e^{-\lambda} - \left(\frac{1}{\lambda^4} + \frac{2}{3\lambda^5} \right) e^{-\lambda/2} + \frac{5}{3\lambda^5} \right\}. \tag{34}$$

We next outline the evaluation of the second term in equation (24) in which $\mathbf{X}_1 \in H$ and $\mathbf{X}_2 \in B$. This evaluation starts by deriving another set of seven expressions similar to equations (26–32) above. Integrating these equations individually over x_2, y_2 we get

$$\int_{X_2 \in B} t(\mathbf{x}_1, \mathbf{x}_2) f_{X_2|X_1 \in H}(\mathbf{x}_2)\, d\mathbf{x}_2 = \alpha \left\{ \frac{1}{\lambda^3} (e^{-\lambda(x_1 - y_1)} + 4e^{-\lambda/2(x_1 + y_1)}) - \frac{e^{-\lambda}}{\lambda} x_1^2 \right.$$

$$\left. + \left(2\left(\frac{1}{\lambda} + \frac{1}{\lambda^2} \right) e^{-\lambda} + \frac{1}{\lambda^2} \right) x_1 - \frac{e^{-\lambda}}{\lambda} y_1^2 + \left(\frac{1}{\lambda} + \frac{1}{\lambda^2} \right) e^{-\lambda} y_1 - 3\left(\frac{1}{2\lambda} + \frac{1}{\lambda^2} + \frac{1}{\lambda^3} \right) e^{-\lambda} - \frac{2}{\lambda^3} \right\}. \tag{35}$$

Integrating equation (35) over $\mathbf{X}_1 \in H$ provides for

$$\int_{X_1 \in H} \int_{X_2 \in B} t(\mathbf{x}_1, \mathbf{x}_2) f_{X_2|X_1 \in H}(\mathbf{x}_2)\, d\mathbf{x}_2 f_{X_1}(\mathbf{x}_1)\, d\mathbf{x}_1 = \alpha^2 \left\{ \left(\frac{1}{3\lambda^2} + \frac{3}{2\lambda^3} + \frac{6}{\lambda^4} + \frac{55}{3\lambda^5} \right) e^{-2\lambda} \right.$$

$$\left. - \left(\frac{1}{\lambda^3} - \frac{1}{\lambda^4} + \frac{22}{\lambda^5} \right) e^{-3/2\lambda} - \left(\frac{1}{\lambda^3} + \frac{1}{\lambda^4} - \frac{3}{\lambda^5} \right) e^{-\lambda} + \left(\frac{1}{\lambda^4} + \frac{2}{3\lambda^5} \right) e^{-\lambda/2} \right\}. \tag{36}$$

Inserting equations (34) and (36) into equation (24) we finally get

$$E_3[T_2] = \alpha^2 \left\{ \left(\frac{1}{3\lambda^2} + \frac{1}{\lambda^3} + \frac{4}{\lambda^4} + \frac{11}{\lambda^5} \right) e^{-2\lambda} - \frac{32}{3\lambda^5} e^{-3/2\lambda} - \left(\frac{2}{\lambda^3} + \frac{2}{\lambda^5} \right) e^{-\lambda} + \frac{5}{3\lambda^5} \right\}. \tag{37}$$

5.2. Computation of $E_4[T_2]$

Getting $E_4[T_2]$ we partition region A (Fig. 3) into regions a, b and c as shown in Fig. 6. When $\mathbf{X}_1 \in B$ and $\mathbf{X}_2 \in A$ we use the property $K \cup H = B$ to write

$$E_4[T_2] = \int_{X_1 \in K} \int_{X_2 \in A} t(\mathbf{x}_1, \mathbf{x}_2) f_{X_2|X_1 \in K}(\mathbf{x}_2)\, d\mathbf{x}_2 f_{X_1}(\mathbf{x}_1)\, d\mathbf{x}_1$$

$$+ \int_{X_1 \in H} \int_{X_2 \in A} t(\mathbf{x}_1, \mathbf{x}_2) f_{X_2|X_1 \in H}(\mathbf{x}_2)\, d\mathbf{x}_2 f_{X_1}(\mathbf{x}_1)\, d\mathbf{x}_1. \tag{38}$$

Using the partition of A to three subregions as in Fig. 6 leads to the following composition of the first term in equation (38)

$$\int_{X_1 \in K} \int_{X_2 \in A} t(\mathbf{x}_1, \mathbf{x}_2) f_{X_2|X_1 \in K}(\mathbf{x}_2)\, d\mathbf{x}_2 f_{X_1}(\mathbf{x}_1)\, d\mathbf{x}_1$$

$$= \int_{X_1 \in K} \left(\sum_{S=a}^{c} \int_{X_2 \in S} t(\mathbf{x}_1, \mathbf{x}_2) f_{X_2}(\mathbf{x}_2)\, d\mathbf{x}_2 \right) f_{X_1}(\mathbf{x}_1)\, d\mathbf{x}_1. \tag{39}$$

Now we can evaluate

$$\int_{X_2 \in S} t(\mathbf{x}_1, \mathbf{x}_2) f_{X_2}(\mathbf{x}_2)\, d\mathbf{x}_2$$

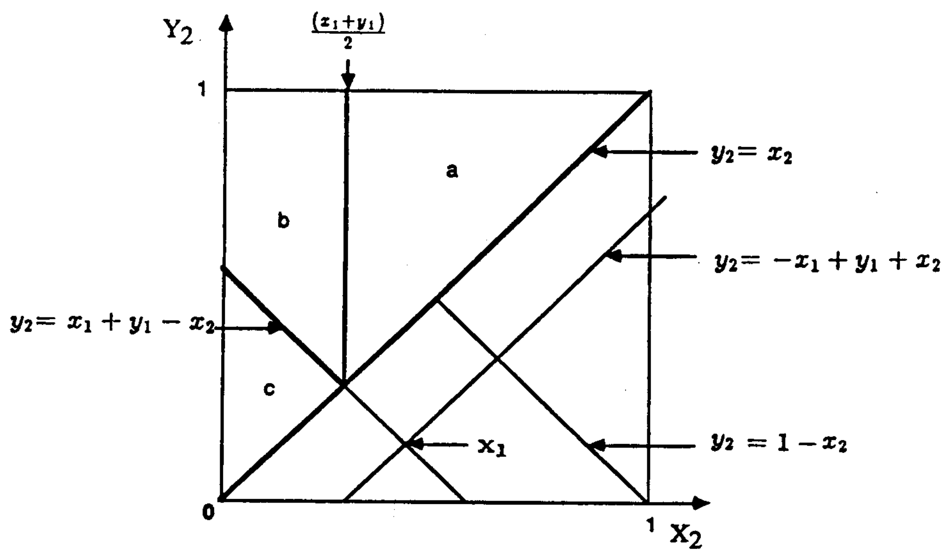

Fig. 6. Partition of region A.

for $S = a, b$, and c. It is done using the following three expressions:

$$\int_{\mathbf{X}_2 \in a | \mathbf{X}_1 \in K} t(\mathbf{x}_1, \mathbf{x}_2) f_{\mathbf{X}_2}(\mathbf{x}_2)\, \mathrm{d}\mathbf{x}_2 = \int_{(x_1+y_1)/2}^{1} \int_{x_2}^{1} (y_2 - y_1)\alpha\, \mathrm{e}^{-\lambda y_2}\, \mathrm{d}y_2\, \mathrm{d}x_2 \tag{40}$$

$$\int_{\mathbf{X}_2 \in b | \mathbf{X}_1 \in K} t(\mathbf{x}_1, \mathbf{x}_2) f_{\mathbf{X}_2}(\mathbf{x}_2)\, \mathrm{d}\mathbf{x}_2 = \int_{0}^{(x_1+y_1)/2} \int_{x_1+y_1-x_2}^{1} (y_2 - y_1)\alpha\, \mathrm{e}^{-\lambda y_2}\, \mathrm{d}y_2\, \mathrm{d}x_2 \tag{41}$$

$$\int_{\mathbf{X}_2 \in c | \mathbf{X}_1 \in K} t(\mathbf{x}_1, \mathbf{x}_2) f_{\mathbf{X}_2}(\mathbf{x}_2)\, \mathrm{d}\mathbf{x}_2 = \int_{0}^{(x_1+y_1)/2} \int_{x_2}^{x_1+y_1-x_2} (x_1 - x_2)\alpha\, \mathrm{e}^{-\lambda y_2}\, \mathrm{d}y_2\, \mathrm{d}x_2. \tag{42}$$

Summing up individual integrals of equations (40–42) we obtain

$$\int_{\mathbf{X}_2 \in A | \mathbf{X}_1 \in K} t(\mathbf{x}_1, \mathbf{x}_2) f_{\mathbf{X}_2|\mathbf{X}_1 \in K}(\mathbf{x}_2)\, \mathrm{d}\mathbf{x}_2 = \alpha\left\{ -\frac{1}{\lambda^3}\mathrm{e}^{-\lambda(x_1+y_1)} + \frac{4}{\lambda^3}\mathrm{e}^{-\lambda/2(x_1+y_1)} + \frac{1}{\lambda^2}x_1 \right.$$
$$\left. + \left(\frac{1}{\lambda} + \frac{1}{\lambda^2}\right)\mathrm{e}^{-\lambda} y_1 - \left(\frac{1}{\lambda} + \frac{2}{\lambda^2} + \frac{2}{\lambda^3}\right)\mathrm{e}^{-\lambda} - \frac{1}{\lambda^3}\right\}. \tag{43}$$

Integrating equation (43) over $\mathbf{X}_1 \in K$ results in

$$\int_{\mathbf{X}_1 \in K} \int_{\mathbf{X}_2 \in A | \mathbf{X}_1 \in K} t(\mathbf{x}_1, \mathbf{x}_2) f_{\mathbf{X}_2|\mathbf{X}_1 \in K}(\mathbf{x}_2)\, \mathrm{d}\mathbf{x}_2 f_{\mathbf{X}_1}(\mathbf{x}_1)\, \mathrm{d}\mathbf{x}_1$$
$$= \int_0^{1/2} \int_{y_1}^{1-y_1} \left(\sum_{S=a}^{c} \int_{\mathbf{X}_2 \in S} t(\mathbf{x}_1, \mathbf{x}_2) f_{\mathbf{X}_2}(\mathbf{x}_2)\, \mathrm{d}\mathbf{x}_2 \right) \alpha\, \mathrm{e}^{-\lambda x_1}\, \mathrm{d}x_1\, \mathrm{d}y_1$$
$$= \alpha^2 \left\{ -\left(\frac{1}{\lambda^3} + \frac{3}{\lambda^4} + \frac{7}{2\lambda^5}\right)\mathrm{e}^{-2\lambda} + \left(\frac{1}{\lambda^3} + \frac{3}{\lambda^4} + \frac{22}{3\lambda^5}\right)\mathrm{e}^{-3/2\lambda} \right.$$
$$\left. - \left(\frac{1}{\lambda^3} + \frac{4}{\lambda^5}\right)\mathrm{e}^{-\lambda} - \left(\frac{1}{\lambda^4} + \frac{2}{\lambda^5}\right)\mathrm{e}^{-\lambda/2} + \frac{13}{6\lambda^5} \right\}. \tag{44}$$

Expression (44) evaluates the first term in equation (38). Getting the second term in equation (38) where $\mathbf{X}_1 \in H$ and $\mathbf{X}_2 \in A$ we follow a similar pattern as in equations (40–42). This effort results in

$$\int_{\mathbf{X}_2 \in A | \mathbf{X}_1 \in H} t(\mathbf{x}_1, \mathbf{x}_2) f_{\mathbf{X}_2 | \mathbf{X}_1 \in H}(\mathbf{x}_2)\, d\mathbf{x}_2 = \alpha \left\{ \frac{4}{\lambda^3} e^{-\lambda/2(x_1+y_1)} - \frac{e^{-\lambda}}{2\lambda}(x_1+y_1)^2 + \left(\left(\frac{1}{\lambda}+\frac{1}{\lambda^2}\right)e^{-\lambda}+\frac{1}{\lambda^2}\right)x_1 + \left(\frac{2}{\lambda}+\frac{2}{\lambda^2}\right)e^{-\lambda} y_1 - 3\left(\frac{1}{2\lambda}+\frac{1}{\lambda^2}+\frac{1}{\lambda^3}\right)e^{-\lambda} - \frac{1}{\lambda^3} \right\}. \quad (45)$$

Integrating equation (45) over $\mathbf{X}_1 \in H$ results in the following expression for the second term in equation (38)

$$\int_{\mathbf{X}_1 \in H} \int_{\mathbf{X}_2 \in A | \mathbf{X}_1 \in H} t(\mathbf{x}_1, \mathbf{x}_2) f_{\mathbf{X}_2 | \mathbf{X}_1 \in H}(\mathbf{x}_2)\, d\mathbf{x}_2 f_{\mathbf{X}_1}(\mathbf{x}_1)\, d\mathbf{x}_1 = \alpha^2 \left\{ \left(\frac{2}{3\lambda^2}+\frac{3}{\lambda^3}+\frac{8}{\lambda^4}+\frac{14}{\lambda^5}\right)e^{-2\lambda} - \left(\frac{1}{\lambda^3}+\frac{3}{\lambda^4}+\frac{18}{\lambda^5}\right)e^{-3/2\lambda} - \left(\frac{1}{\lambda^3}+\frac{2}{\lambda^4}-\frac{2}{\lambda^5}\right)e^{-\lambda} + \left(\frac{1}{\lambda^4}+\frac{2}{\lambda^5}\right)e^{-\lambda/2} \right\}. \quad (46)$$

Inserting equations (44) and (46) into equation (38) leads to

$$E_4[T_2] = \alpha^2 \left\{ \left(\frac{2}{3\lambda^2}+\frac{2}{\lambda^3}+\frac{5}{\lambda^4}+\frac{21}{\lambda^5}\right)e^{-2\lambda} - \frac{32}{3\lambda^5}e^{-3/2\lambda} - \left(\frac{2}{\lambda^3}+\frac{2}{\lambda^4}+\frac{2}{\lambda^5}\right)e^{-\lambda} + \frac{13}{6\lambda^5} \right\}. \quad (47)$$

Inserting equations (37) and (47) into equation (23) provides for the expected value of interleaving time

$$E[T_2] = 2\alpha^2 \left\{ \left(\frac{1}{\lambda^2}+\frac{3}{\lambda^3}+\frac{9}{\lambda^4}+\frac{43}{2\lambda^5}\right)e^{-2\lambda} - \frac{64}{3\lambda^5}e^{-3/2\lambda} - \left(\frac{4}{\lambda^3}+\frac{2}{\lambda^4}+\frac{4}{\lambda^5}\right)e^{-\lambda} + \frac{23}{6\lambda^5} \right\}. \quad (48)$$

Note that for the special case of random storage allocation, we can use $\lim_{\lambda\to 0} E[T_2] = 7/15$. This conforms with the earlier results of Graves *et al.* [5]. To find the above limit we restructured equation (48) into a single term fractional form. This form results in $\frac{0}{0}$ when $\lambda \to 0$. Getting the desired bound leads to five consecutive applications of L'Hospital's rule.

6. EXPECTED VALUE OF DUAL COMMAND

Inserting equations (10) and (48) into equation (7) gives

$$E[DC] = 2\alpha^2 \left\{ \left(\frac{1}{\lambda^2}+\frac{3}{\lambda^3}+\frac{9}{\lambda^4}+\frac{43}{2\lambda^5}\right)e^{-2\lambda} - \frac{64}{3\lambda^5}e^{-3/2\lambda} - \left(\frac{4}{\lambda^3}+\frac{2}{\lambda^4}+\frac{4}{\lambda^5}\right)e^{-\lambda} + \frac{23}{6\lambda^5} \right\} + \frac{4e^{\lambda}-2\lambda^2-4\lambda-4}{\lambda(e^{\lambda}-\lambda-1)}. \quad (49)$$

Note that for the uniform case we use $\lim_{\lambda\to 0} E[DC] = 9/5$. This also conforms with the earlier result of Graves *et al.* [5] for the special case of random item allocation.

7. NUMERICAL EXAMPLES

This section presents a comparative evaluation of random storage ($\lambda = 0$) with the full turnover based storage allocation policies.

The fourth and the sixth columns in Table 1 represent the potential savings in the expected cycle times when full turnover allocation policy is implemented instead of random allocation. For example, in the case of 80/30 the expected savings are 46.28 and 41.98% for single command and for dual command, respectively.

The results indicates that significant time savings can be realized using full turnover rate rather than random storage assignment. It is also clear that these benefits become more pronounced with the increase in the concentration of the fast moving items, and that slightly greater savings are expected in the single command case.

Table 1. The expected cycle times and potential savings for some turnover rates

Item allocation	λ	Expected cycle times and potential savings			
		$E[SC]$	$100\left(1-\frac{E[SC]}{E[SC \mid \text{Random}]}\right)\%$	$E[DC]$	$100\left(1-\frac{E[DC]}{E[DC \mid \text{Random}]}\right)\%$
Random	0.0000	1.3333	—	1.8000	—
60/30	2.8992	0.9731	27.02	1.3818	23.23
70/30	3.9119	0.8490	36.30	1.2226	32.08
80/30	5.1358	0.7162	46.28	1.0440	41.98
90/30	6.9489	0.5622	57.82	0.8275	54.02

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