

# Common Due Date Assignment to Minimize Total Penalty for the One Machine Scheduling Problem

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We consider an  $n$  job, one machine scheduling problem in which all jobs have a common due date. The objective is to determine the optimal value of this due date and an optimal sequence to minimize a total penalty function. This penalty function is based on the due-date value and on the earliness or the lateness of each job in the selected sequence. We present a polynomial bound scheduling algorithm for the solution of this problem along with the proof of optimality, a numerical example and discuss some extensions.

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**I**N THIS PAPER, we consider the basic one machine sequencing problem under the common assumptions listed in Baker [1974]. All jobs have a common (but unknown) due date. The objective is to find an optimal value of the due date *and* an optimal sequence which minimizes the total penalty based on the due date value and the earliness or tardiness of each job. We will discuss possible extensions of the problem and a case involving distinct due dates.

Job scheduling and sequencing can be approached from either the viewpoint of the shop that will process the job or the customer placing the order for the job. If the shop viewpoint is taken, the objective of the schedule determination is likely to be related to the minimization of one or more cost factors. When approached from the customer viewpoint, the objective likely will be related to the due date. Due dates are usually highly important in the schedule determination except perhaps when the production run is made for inventory.

The most elementary form of due dates is that of constant lead time. With this method the due date is simply the time of order plus a constant lead time. If the shop status is relatively stable, all jobs may be given due dates based upon the constant lead time regardless of job content.

Most studies involving scheduling with due dates have treated due dates as exogenous decisions. Conway [1965] was one of the first to study

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common lead time due dates. Other job shop studies using various due dates have been reported in the literature (see for example, Allen et al. [1975], Eilon and Chaudhary [1976], Heard [1970], Jones [1973], Nuttle and Woolfolk [1979], Reinitz [1963], Weeks [1979], and Weeks and Fryer [1977]). Some of these studies include specified common due dates for the one machine problem with minimum total tardiness criterion.

Of the few studies analyzing due date selection procedures, that of Baker and Bertrand [1980] is the most recent and directly related to the present one. They considered (among other things) a static one machine problem. One of the due date assignment procedures involved in their study was the common due date. Their objective, however, was only to find the lowest value of due date such that no job would be tardy.

Weeks and Fryer simulated job shop problems to determine good due dates. Their cost function included linear tardiness costs, linear earliness costs, linear flow time costs, linear labor transfer costs, and nonlinear due date costs.

The per unit costs involved in this paper (due date, earliness, and tardiness) are all linear. Linear cost functions present a case that is more tractable than that occurring with nonlinear costs. The insight gained from the linear model may be useful when approaching the nonlinear model. Also it is likely that in practice the estimation of costs may introduce more inaccuracies than those occurring with the assumption of linear costs. All of these costs can be regarded as opportunity costs. Two of the costs, due date and tardiness, are related to customer objectives while the earliness cost can be viewed as an important element that concerns the production shop. The three costs are a function of the due date and hence should be considered when a constant lead time due date is established.

## 1. PROBLEM FORMULATION

Let  $N$  denote the set of  $n$  jobs and let  $t_i$  denote the processing time of job  $i$  ( $i \in N$ ). We assume that the job labeling is such that  $t_1 \leq t_2 \leq \dots \leq t_n$ ; i.e., sequence 1, 2,  $\dots$ ,  $n$  represents the SPT sequence. All jobs are assumed to have a common due date  $d$ .

If  $C_i$ ,  $E_i$ , and  $T_i$  respectively denote the completion time, the earliness and the tardiness of job  $i$ , then

$$E_i = \max(0, d - C_i) \quad \text{and} \quad T_i = \max(0, C_i - d).$$

Let  $\pi$  denote the set of  $n!$  sequences, with  $\sigma$  denoting an arbitrary sequence and let  $R$  denote the set of nonnegative real numbers. The total penalty  $f(d, \sigma)$  associated with a specified value of  $d$  and a specified sequence  $\sigma$  is given by

$$f(d, \sigma) = \sum_{i=1}^n (P_1 d + P_2 E_{[\sigma_i]} + P_3 T_{[\sigma_i]}),$$

where  $P_1$ ,  $P_2$  and  $P_3$  are nonnegative constants and the subscript  $[i]$  is used to denote the job in position  $i$  for sequence  $\sigma$ .  $P_1$  represents the due date assignment cost per unit time;  $P_2$  and  $P_3$  are the earliness and tardiness costs per unit time, respectively. We can now write the objective function as

$$\min f(d, \sigma), \quad d \in R \quad \text{and} \quad \sigma \in \pi.$$

## 2. PRELIMINARY ANALYSIS

In this section we present many elementary results and some lemmas. For brevity, only the proofs of lemmas are presented. An algorithm based on the analysis is presented in the next section.

RESULT 1.  $d^* \leq C_{[n]} = \sum_{i=1}^n t_i$ ,  $i \in N$ , where  $d^*$  denotes the optimal due date.

RESULT 2. If  $P_1 \geq P_3$ ,  $d^* = 0$  and SPT is optimal.

We now prove the following two lemmas related to the optimal due date value. Throughout the remainder of this section we assume  $P_1 < P_3$ .

LEMMA 1. For any specified sequence  $\sigma$ , there exists an optimal value of  $d$  which coincides with the completion time of one of the jobs in  $\sigma$ .

*Proof.* Suppose  $d < C_{[1]}$ . Then all jobs are tardy. If the due date is increased to  $C_{[1]}$ , the total penalty will increase by  $n(P_1 - P_3)(C_{[1]} - d)$ . This quantity is nonpositive.

Now suppose  $C_{[i]} < d < C_{[i+1]}$ ,  $i = 1, 2, \dots, n-1$ . Let  $x = d - C_{[i]}$  and let  $y = C_{[i+1]} - d$  so that  $x > 0$  and  $y > 0$ . If the due date is changed to  $C_{[i]}$ , the new penalty is given by

$$f(C_{[i]}, \sigma) = f(d, \sigma) + x(n(P_3 - P_1) - i(P_2 + P_3)).$$

Similarly, if the due date is changed to  $C_{[i+1]}$ , the penalty is given by

$$f(C_{[i+1]}, \sigma) = f(d, \sigma) - y(n(P_3 - P_1) - i(P_2 + P_3)).$$

Obviously,  $f(C_{[i]}, \sigma) \leq f(d, \sigma)$  if  $(n(P_3 - P_1) - i(P_2 + P_3)) \leq 0$  and  $f(C_{[i+1]}, \sigma) < f(d, \sigma)$  otherwise.

It is now clear that an optimal value of  $d$  coincides with the completion time of a job.

LEMMA 2. For any specified sequence  $\sigma$ , there exists an optimal due date equal to  $C_{[K]}$ , where  $K$  is the smallest integral value greater than or equal to  $n(P_3 - P_1)/(P_2 + P_3)$ .

*Proof.* From Lemma 1 we know that an optimal due date value can coincide with the completion time of some job. Let this be the job in position  $K$ , where  $1 \leq K \leq n$ . The total penalty is then equal to

$f(C_{[K]}, \sigma)$ . Since this due date is optimal, we know that for any  $\Delta > 0$ ,

$$f(C_{[K]} + \Delta, \sigma) - f(C_{[K]}, \sigma) \geq 0.$$

The right shift of the due date from  $C_{[K]}$  to  $C_{[K]} + \Delta$  causes an increase in the first two components of the total penalty by at least  $\Delta(nP_1 + KP_2)$  and causes a decrease in the third component by no more than  $\Delta(n - K)P_3$ .

Therefore,  $\Delta(nP_1 + KP_2 - (n - K)P_3) \geq 0$ ,

or,  $K \geq n(P_3 - P_1)/(P_2 + P_3)$ .

Also,  $f(C_{[K]} - \Delta, \sigma) - f(C_{[K]}, \sigma) \geq 0$ ,

or, the left shift of the due date results in

$$K - 1 \leq n(P_3 - P_1)/(P_2 + P_3).$$

Since  $K$  is an integer, the proof of Lemma 2 follows immediately.

From the discussion thus far, it is clear that for any sequence, exactly  $K$  jobs will be nontardy ( $K = 0$  if  $P_1 \geq P_3$ ). The total penalty is equal to

$$f(C_{[K]}, \sigma) = \sum_{i=1}^n (P_1 d + P_2 E_{[i]} + P_3 T_{[i]}).$$

Substituting  $C_{[K]} = t_{[1]} + t_{[2]} + \dots + t_{[K]}$  and simplifying, we get,

$$\begin{aligned} f(C_{[K]}, \sigma) &= \sum_{j=1}^K (nP_1 + (j-1)P_2)t_{[j]} + \sum_{j=K+1}^n P_3(n+1-j)t_{[j]} \\ &= \sum_{j=1}^n \gamma_j t_{[j]}, \quad \text{where } \gamma_j \text{ is equal to } nP_1 + (j-1)P_2 \end{aligned}$$

for  $j \leq K$  and is equal to  $P_3(n+1-j)$  for  $j > K$ .

To find the optimal sequence, we need to find the minimal penalty among all  $\sigma \in \pi$ . The term  $\gamma_j$  may be viewed as the positional penalty for sequence position  $j$ .

**RESULT 3.** *Quantity  $(\sum_{j=1}^n \gamma_j t_{[j]})$  is minimized by matching the smallest value of  $\gamma$  with the largest value of  $t$ , the next larger value of  $\gamma$  with the next smaller value of  $t$ , and so on.*

The results and the lemmas discussed above are incorporated in the algorithm as described below.

### 3. OPTIMAL SOLUTION PROCEDURE

The algorithm presented here consists of two phases. Phase 1 finds the number of nontardy jobs ( $K$ ). Phase 2 calculates positional penalties  $\gamma$ , the optimal sequence, and the optimal due date ( $d^*$ ).

#### Phase 1

*Step 1.1.* Set  $K' \leftarrow n(P_3 - P_1)/(P_2 + P_3)$ .

Step 1.2. Check if  $K' > 0$ .

If YES: go to Step 1.3

If NO: set  $d^* \leftarrow 0$ .

SPT sequence is optimal.

STOP.

Step 1.3. Check if  $K'$  is an integer.

If YES: set  $K \leftarrow K'$ .

Proceed to Phase 2.

If NO: set  $K$  equal to the smallest integer value  $> K'$ .

## Phase 2

Step 2.1. Label position  $j$  ( $1 \leq j \leq n$ ) as

$$\gamma_j = \begin{cases} nP_1 + (j-1)P_2, & 1 \leq j \leq K \\ (n+1-j)P_3, & K+1 \leq j \leq n. \end{cases}$$

Step 2.2. Rank the positional labels  $\gamma_j$  in descending order of magnitude such that the largest  $\gamma_j$  is ranked 1 and the smallest  $\gamma_j$  is ranked  $n$ . Break ties arbitrarily.

Step 2.3. Obtain the optimal sequence such that job  $i$  is scheduled in position  $j$  corresponding to  $\gamma_j$  ranked in position  $i$ .

Step 2.4. Set  $d^* \leftarrow (t_{[1]} + t_{[2]} + \dots + t_{[K]})$ . STOP.

## Numerical Example

Given seven jobs with  $t_1 = 3$ ,  $t_2 = 4$ ,  $t_3 = 6$ ,  $t_4 = 9$ ,  $t_5 = 14$ ,  $t_6 = 18$ , and  $t_7 = 20$ . The penalties are  $P_1 = 5$ ,  $P_2 = 11$  and  $P_3 = 18$ .

From Phase 1 of the algorithm, we get  $K' = 3.13$  and thus  $K = 4$ . The seven positional labels and their ranks are as indicated below.

Position $j$	1	2	3	4	5	6	7
$\gamma_j$	35	46	57	68	54	36	16
Rank $i$	6	4	2	1	3	5	7

Optimal sequence: 6, 4, 2, 1, 3, 5, 7

$$d^* = t_6 + t_4 + t_2 + t_1 = 34.$$

One can easily verify that the total penalty is 2,664 units.

## 4. FURTHER ANALYSIS AND EXTENSIONS

Consider  $\gamma_j$  defined earlier. We now define

$$\alpha_j = nP_1 + (j-1)P_2, \quad \text{for } j = 1, 2, \dots, n,$$

and 
$$\beta_j = (n+1-j)P_3, \quad \text{for } j = 1, 2, \dots, n.$$

Figure 1 gives a plot of  $\alpha_j$  and  $\beta_j$  vs.  $j$ . (For convenience we have used data from the numerical example.) It is easy to see that

$$\gamma_j = \min(\alpha_j, \beta_j) \quad \text{for } j = 1, 2, \dots, n.$$

In fact, the number of nontardy jobs ( $K$ ) can be determined from the  $\alpha_j$  and  $\beta_j$  values, for

$$\alpha_j \leq \beta_j \quad \text{for } j = 1, 2, \dots, K$$

and

$$\alpha_j \geq \beta_j \quad \text{for } j = K + 1, K + 2, \dots, n.$$

We can observe that the positional penalties  $\gamma$  have a pyramidal structure while the sequence generated by the algorithm has an inverse pyramidal (LPT-SPT) structure (assuming that in Step 2.2 of the algo-

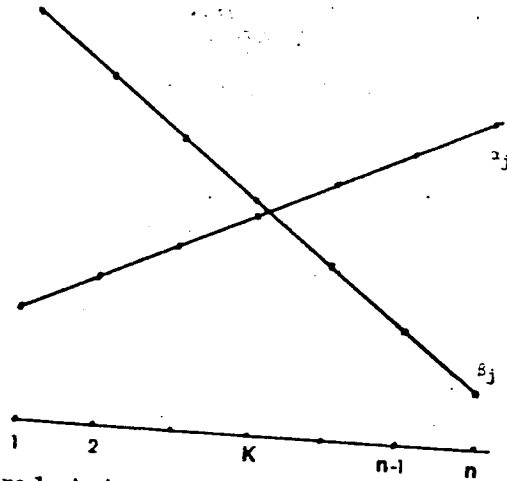


Figure 1. A plot of  $\alpha, \beta$  values vs. sequence positions.

rithm, we break ties to maintain the LPT-SPT structure). The inverse pyramidal structure of the optimal sequence is to be expected since the early jobs must be in the LPT order to minimize the earliness penalty and the late jobs must be in the SPT order to minimize the tardiness penalty.

It may be noticed that if  $P_1 \geq P_3$ ,  $\alpha_j \geq \beta_j$  for  $j = 1, 2, \dots, n$ , then  $K = 0$ , (i.e.,  $d^* = 0$ ). Notice that if  $P_2 = 0$  (no earliness cost), all  $\alpha_j$  values are equal. If  $P_1 = 0$  (earliness-tardiness problem),  $\alpha_1 = 0$ .

Now we consider some possible extensions. Preliminary analysis showed that changing penalty functions to the nonlinear ones or allowing nonzero ready times do not lend themselves to polynomially bound solution procedures. The following three cases include two solvable ones and one which may not have a closed form solution.

(i) *Addition to the flow time penalty*

Suppose we denote the flow time of sequence  $\sigma$  by  $F_\sigma$ . Let the total penalty be given by the total penalty defined in Section 1 plus  $P_4 F_\sigma$ , where  $F_\sigma = \sum C_{[i]}$ . It is easy to determine the optimal sequence and  $d^*$  as follows.

First we observe that Result 1 is still valid. Result 2 is also true since the total flow time is minimized by the SPT sequence. Similarly Lemmas 1 and 2 apply to the present case as these lemmas imply a specified sequence  $\sigma$ . Thus the number of nontardy jobs is still equal to  $K$  and

$$d^* = t_{[1]} + t_{[2]} + \dots + t_{[K]}.$$

The optimal sequence however, may be different since  $\gamma_j$  values are affected by the flow time penalty. Writing

$$F_\sigma = nt_{[1]} + (n-1)t_{[2]} + \dots + t_{[n]},$$

it is easy to verify that

$$\alpha_j = nP_1 + (j-1)P_2 + (n+1-j)P_4 \quad \text{for } j = 1, 2, \dots, n,$$

$$\text{and } \beta_j = (n+1-j)P_3 + (n+1-j)P_4 \quad \text{for } j = 1, 2, \dots, n.$$

This means that the positional penalty  $\gamma_j$  increases by  $(n+1-j)P_4$  for each position  $j$ . Result 3 can then be applied to obtain the optimal sequence.

It is interesting to note that the optimal sequence may no longer be of the inverse pyramidal shape. In fact, if  $P_4 > P_2$ , the flow time penalty will dominate the earliness penalty and the optimal sequence becomes SPT (even though  $P_1 < P_3$  and  $K > 0$ ).

(ii) *Distinct due dates with acceptable lead time*

Consider the original problem discussed in Section 1, except each job  $i$  can have a distinct due date  $d_i$ . Assume that the customers expect a reasonable nonnegative lead time value "A" for the due date so that the lead time penalty  $P_1$  is applied only if the due date exceeds "A." (Weeks and Fryer used the concept of "A" with their nonlinear due date cost function.) The total penalty function now becomes

$$(P_1 A_i + P_2 E_i + P_3 T_i)$$

where  $A_i = \max(0, d_i - A)$ .

This problem is solvable for any  $A \geq 0$ . Seidman et al. [1981] have shown the following.

- (a) SPT sequence is always optimal
- (b) If  $P_1 \leq P_3$ , set  $d_i^* = C_i$  in the SPT sequence, otherwise set  $d_i^* = \min(A, C_i \text{ in the SPT})$ .

Note that no job will be early in the optimum sequence (SPT) and hence the solution is independent of the value of  $P_2$ .

(iii) *Common due date, acceptable lead time*

Consider the original problem with the nonnegative lead time "A." The first component of the total penalty function now becomes

$$P_1 \cdot \max(0, d - A).$$

If  $A = 0$ , the problem reduces to the original problem discussed earlier. For  $A > 0$ , however, the problem may not have a closed form solution. To illustrate this point, consider the following case. Suppose  $P_1 > P_3 > P_2 > 0$ . One can easily prove that  $d^* = A$ . However, the number of nontardy jobs can no longer be determined independently of the processing time values. Even if an optimal sequence has an inverse pyramid shape, there are  $2^{n-1}$  potential optimal sequences.

Special cases of this problem may be easily solved. For example, consider  $P_2 = 0$  and  $P_1 > P_3$ . The solution is given by  $d^* = A$  with the SPT sequence being optimal.

In certain specific instances, one may be able to find the optimal solution quickly even if there is no closed form solution in general. For example, consider  $P_1 < P_3$ . Suppose we assume that  $P_1 = 0$  and solve the problem instance using the algorithm presented earlier. If  $d^* < A$ , we can easily show that the solution is optimal.

## 5. CONCLUDING REMARKS

The algorithm presented in Section 3 is polynomially bounded and is of the order  $O(n \log(n))$ . Hence the problem belongs to class  $P$ . It is interesting to note that the determination of  $K$ , the number of nontardy jobs, does not require the values of the processing times of the  $n$  jobs.

Additional extensions to those developments include nonlinear continuous cost functions, set up cost functions and multiple machines, to name a few. While these cases have desirable elements of realism, they likely will be found intractable from an optimization standpoint.

From the practical point of view, the conclusion here is that the optimum due date can be selected if one can anticipate completion times of different jobs. A similar conclusion was drawn by Weeks in his multimachine job shop simulation study.

Finally, it must be pointed out that weighted objective functions are hard to implement in practice for at least two reasons. First, it is difficult to estimate proper weights. Second, the penalties implied by these weights is notational and may not easily be used in the calculation of savings resulting from good scheduling.

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