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Optimal assignment of due-dates for a single processor scheduling problem

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Given processing times of n jobs on a single machine with penalties for earliness and tardiness and penalties associated with assignment of due-dates, the objective is to select optimal due-dates and optimal sequence. Scheduling procedure for the solution of this problem is presented along with proof of optimality and illustrative numerical examples.

Introduction

Consider the following well known problem from the scheduling literature. There are n independent jobs to be processed on a single processor. All jobs are available at time zero; only one job is processed at any given time, and no preemption is permitted. Once processing begins on the first job, processing continues uninterrupted until all jobs are completed§.

This fundamental scheduling model has received much attention in the literature over the years. Smith (1956) proved that the mean flow time is minimized by the SPT sequence. Moore (1968) proposed an elegant procedure to minimize the number of tardy jobs when an arbitrary due-date is specified for each job. Many researchers, including Emmons (1969), Srinivasan (1971), Rinnooy Kan *et al.* (1975), Baker and Schrage (1978), have considered the problem of finding a sequence that minimizes the total tardiness. In these studies and in many others, the due-date (i.e., the specified delivery date) for every job is assumed to represent an exogenous decision. A few studies related to the systematic assignment of due-dates have appeared in the literature. Examples are: Heard (1970), Eilon and Chowdhury (1976), Weeks and Fryer (1977) and Weeks (1979).

In this paper it is assumed that one is free to assign any due-date value (possibly distinct) for each job. Three types of penalties are considered: (*a*) 'lead-time' penalty dependent upon the specific due-date assigned to each individual job, (*b*) earliness penalty proportional to the earliness of each job, and (*c*) tardiness penalty proportional to the tardiness of each job.

The particular component penalties used in this paper are primarily opportunity costs. The lead time penalty represents the potential loss of sales associated with quoting long lead time to the customers. If the promised delivery date is within this lead time, there is no penalty. However, each additional unit of quoted delivery date beyond the reasonable lead time causes a lead time penalty (Jones 1973, Weeks and Fryer 1977). The earliness penalty is associated with holding costs of jobs which have been completed before their respective due-dates.

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§ Baker (1974) provides a complete set of common assumptions for the one machine problem.

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Two questions must be answered here. First, what are the 'optimal' values of due-dates? Second, which production sequence is optimal? Answers to these two questions are proposed in this paper for the problem defined in the following section.

Problem definition

Let N be the set of n given jobs and let t_i denote processing time of job $i \in N$. Without loss of generality, all jobs are assumed to be numbered in an ascending order of processing times so that $t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n$. This will result in $1, 2, 3, \ldots, n$ as the SPT sequence. Let d_i be the due-date of job i. Define $A_i = \max(0, d_i - A)$; A represents the lead time that customers consider to be reasonable and expected; A_i represents the amount of time the assigned due-date for job i exceeds A.

For an arbitrary sequence σ, let C_i, T_i and E_i, respectively, denote the completion time of job i, the tardiness of job i and the earliness of job i. The values of T_i and E_i are given by:

$$T_i = \max(0, C_i - d_i) \text{ and } E_i = \max(0, d_i - C_i)$$

Obviously, both T_i and E_i cannot be positive simultaneously.

Let P_1, P_2 and P_3 be specified per unit notional penalties† such that the total penalty for a sequence σ is given by:

$$\sum_{i \in N} (P_1 A_i + P_2 E_i + P_3 T_i)$$

P_1 is the per unit lead time penalty, P_2 is the per unit holding cost and P_3 is the per unit penalty for tardiness. The objective is to find optimal values $d_i^* (i = 1, 2, \ldots, n)$ *and* an optimal sequence σ^* which minimize the total penalty.

Scheduling procedure and numerical examples

Answers to the two questions raised earlier concerning assignment of due-dates and the sequence for processing the n jobs are given by the following procedure.

Step 1

$$\text{If } P_1 \leqslant P_3, \text{ set } d_i^* = \sum_{j=1}^{i} t_j, i = 1, 2, \ldots, n; \text{ otherwise}$$

$$\text{set } d_i^* = \min \{A, \sum_{j=1}^{i} t_j\}, i = 1, 2, \ldots, n$$

Step 2

Schedule jobs according to the SPT rule (i.e., $1, 2, 3, \ldots, n$).

This procedure provides optimal results for the problem defined above and can easily be used manually. Proofs of optimality are provided in the Appendix. Two numerical examples will illustrate the application of the scheduling procedure.

Example 1

Given six jobs with $t_1 = 5$, $t_2 = 8$, $t_3 = 10$, $t_4 = 12$, $t_5 = 12$ and $t_6 = 15$. The penalties are $P_1 = 15$, $P_2 = 25$, $P_3 = 20$ and $A = 30$.

† For another example of notional costs see Eilon and Chowdhury (1976).

Since $P_1 \leqslant P_3$ the optimal due-dates from Step 1 are: $d_1^* = 5$, $d_2^* = 13$, $d_3^* = 23$, $d_4^* = 35$, $d_5^* = 47$ and $d_6^* = 62$. Total penalty for the SPT sequence (1, 2, 3, 4, 5, 6) is 810; Table 1 gives the component penalties.

Job	Penalties		
	Due-date	Earliness	Tardiness
1	0	0	0
2	0	0	0
3	0	0	0
4	75	0	0
5	255	0	0
6	480	0	0

Table 1. Component penalties for Example 1.

Example 2

Given the same set of jobs with $t_1 = 5$, $t_2 = 8$, $t_3 = 10$, $t_4 = 12$, $t_5 = 12$, and $t_6 = 15$, except $P_1 = 30$, $P_2 = 25$, $P_3 = 20$ and $A = 30$.

Noting that $P_1 > P_3$ the optimal due-dates are $d_1^* = 5$, $d_2^* = 13$, $d_3^* = 23$, $d_4^* = 30$, $d_5^* = 30$ and $d_6^* = 30$. Component penalty costs for the SPT sequence for this example are shown in Table 2; the total penalty cost is 1080 units.

Job	Due-date	Earliness	Tardiness
1	0	0	0
2	0	0	0
3	0	0	0
4	0	0	100
5	0	0	340
6	0	0	640

Table 2. Component penalties for Example 2.

Discussion

This paper examines the due-date selection problem and its interaction with the optimal production sequence. For the single processor model considered in this paper, it has been shown analytically that the optimal due-date assignment is related to the completion time of the respective jobs or to the value of the specified lead time. For certain production environments, this conclusion may be helpful in arriving at appropriate selection of due-dates if one can compute anticipated completion times for different jobs. A recent simulation study of multimachine job shop by Weeks (1979) supports this conclusion.

With this cost structure used here the optimal scheduling procedure will always result in no earliness penalty for any job. Also in each optimal schedule, at most one type of component penalty will be incurred.

Practical scheduling problems are dynamic, multimachine (in most cases), and complex compared to the static, single machine model discussed in this paper. Analysis of single machine models, however, has provided many valuable insights and fundamental properties that may foster improved guidelines for the design of actual scheduling systems; see for example, Elmaghraby (1968).

APPENDIX

Let σ represent any arbitrary sequence and let $t_{[i]}$ represent the processing time of the job in the ith position of this sequence. Similar definition holds for $A_{[i]}$, $C_{[i]}$, $E_{[i]}$, $T_{[i]}$ and $d_{[i]}$. The objective is to determine a set of due-dates $\{d_1, d_2, \ldots, d_n\}$ and a sequence of the n jobs such that the total penalty given by

$$\sum_{i \in N} (P_1 A_i + P_2 E_i + P_3 T_i)$$

is minimized.

Steps 1 and 2 given earlier in the scheduling procedure can be stated in general terms by means of the following theorems.

Theorem 1

For any arbitrary sequence σ, optimal due-dates are given by:

(i) $$d^*_{[i]} = C_{[i]} \text{ if } P_1 \leqslant P_3, \text{ and}$$

(ii) $$d^*_{[i]} = \min\{A, C_{[i]}\} \text{ otherwise.}$$

Theorem 2

The SPT sequence in conjunction with the due-dates described in Theorem 1, yields minimal total penalty cost.

Comments

(1) It may be noted that for any job i in sequence σ, the product of $T_{[i]} E_{[i]} = 0$.

(2) For the SPT sequence, $C_{[i]} = C_i = \sum_{j=1}^{i} t_j$, $i = 1, 2, \ldots, n$.

(3) It may be observed that if the due-date of any job i is increased then $P_1 A_i$ may increase (or remain constant), $P_2 E_i$ may increase (or remain constant) while $P_3 T_i$ may decrease (or remain constant). Reduction in due-date value for job i should produce opposite effects on $P_1 A_i$, $P_2 E_i$ and $P_3 T_i$.

The proofs are presented below.

Proof of Theorem 1

The proof considers two cases.

Case (1) $P_1 \leqslant P_3$

(*a*) Let $d_{[i]} > C_{[i]}$ for job $[i]$ in an arbitrary sequence σ. Obviously, job $[i]$ is nontardy. Now, setting the due-date $d_{[i]} = C_{[i]}$ the tardiness for $[i]$ still remains zero (i.e., $P_3 T_{[i]} = 0$). Also, making $d_{[i]} = C_{[i]}$, reduces the earliness penalty by $P_2(d_{[i]} - C_{[i]})$. Since $A_{[i]} = \max(0, d_{[i]} - A)$, reduction in $d_{[i]}$ can not increase $P_1 A_{[i]}$. Finally, changing of $d_{[i]}$ only affects the contribution of $[i]$ to the total penalty. Hence it is never optimal to have $d_{[i]} > C_{[i]}$ for all $[i]$.

(*b*) Now let $d_{[i]} < C_{[i]}$. In this situation job $[i]$ is tardy. Making $d_{[i]} = C_{[i]}$, the earliness for $[i]$ still remains zero (i.e., $P_2 E_{[i]} = 0$). Also, having $d_{[i]} = C_{[i]}$

reduces the tardiness penalty by $P_3(C_{[i]}-d_{[i]})$. It is possible that $P_1A_{[i]}$ may increase (if $A>C_{[i]}$ there will be no increase). The maximal increase, however, is limited to $P_1(C_{[i]}-d_{[i]})$. Since $P_1 \leqslant P_3$, it is obvious that the total penalty cannot be higher when the due-date of $[i]$ is moved to $C_{[i]}$.

Hence, $d^*_{[i]}=C_{[i]}$ when $P_1 \leqslant P_3$.

Case (2) $P_1 > P_3$

(*a*) Let $d_{[i]} > \min\{A, C_{[i]}\}$ for job $[i]$ in some arbitrary sequence σ.

(i) If $A \leqslant C_{[i]}$, reduction of $d_{[i]}$ to A causes the following penalty changes (Fig. 1 (*a*)). The maximum possible increase in the tardiness penalty will be $P_3(d_{[i]}-A)$ while the earliness penalty cannot increase. The value of P_1A_i will decrease by $P_1(d_{[i]}-A)$ and will become zero. The cumulative effect of all these changes will not increase the total penalty.

(ii) When $C_{[i]}<A$, reduction of $d_{[i]}$ to $C_{[i]}$ will decrease the earliness penalty by $P_2(d_{[i]}-C_{[i]})$. The penalty $P_1A_{[i]}$ will be reduced to zero and the tardiness penalty remains zero (Fig. 1 (*b*)). This results in a net reduction of the total penalty.

From (i) and (ii) it is evident that optimal $d_{[i]} \not> \min\{A, C_{[i]}\}$.

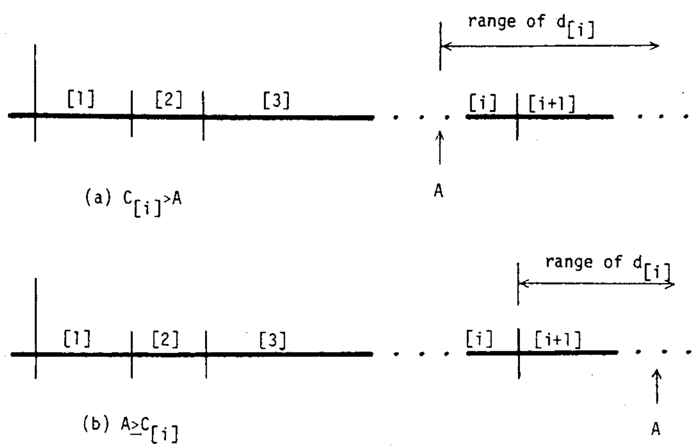

Figure 1. Possible due-date assignment with $d_{[i]} \geqslant \min\{A, C_{[i]}\}$.

(*b*) Suppose $d_{[i]} < \min\{A, C_{[i]}\}$ in an arbitrary sequence σ. The proof that optimal $d_{[i]} \not< \min\{A, C_{[i]}\}$ has been omitted for brevity. In principle, changes of $d_{[i]}$ only affect the contribution of job $[i]$ to the total penalty.

From the results of (*a*) and (*b*) it is clear that in Case (2) optimal $d_{[i]}=\min\{A, C_{[i]}\}$ for $i \in N$. This completes the proof of Theorem 1.

In the following it will be proved that the shortest processing time sequence (SPT) in conjunction with these optimal due-dates minimizes the total penalty.

Proof of Theorem 2

The analysis is partitioned into two cases.

Case (1) $P_1 \leqslant P_3$

Optimality of the SPT sequence can be proved as follows. Since optimal due-dates for all jobs in any sequence σ should be set equal to their respective completion times, no jobs will be either early ($E_i=0$) or tardy ($T_i=0$). The total penalty for any sequence is therefore simply equal to

$$\sum_{i\in N} P_1 A_i$$

$$=P_1 \sum_{i\in N} \max(0, d_i - A)$$

$$=P_1 \sum_{i\in N} \max(0, C_i - A), \text{since } d_i = C_i$$

The objective is therefore to minimize $\sum P_1 \max(0, C_i - A)$ or to minimize $\sum_{i\in N} \max(0, C_i - A)$, since P_1 is a constant.

Baker (1974, p. 36, Theorem 2.10) has presented a well known result, that the SPT sequence minimizes the total tardiness for a one machine problem with a common due-date d. The total tardiness for that common due-date model was computed by

$$\sum_{i\in N} \max(0, C_i - d)$$

If d is replaced by A, the resulting expression corresponds with the above modified objective function. Hence, SPT sequence is optimal for $P_1 \leqslant P_3$.

Case (2) $P_1 > P_3$

According to Theorem 1, optimal $d_{[i]} = \min\{A, C_{[i]}\}$. Thus, for any job i, $d_i \leqslant A$ and $A_i = 0$. Also, $E_i = 0$, since no job will be early. Tardiness penalty will apply to any job with $C_{[i]} > d_i$.

Consider any two consecutive jobs $[i]$ and $[i+1]$ in an arbitrary sequence σ and suppose that $t_{[i]} > t_{[i+1]}$. Interchanging these two jobs can affect only their own tardiness; all other job completion times remain unchanged.

Figure 2 illustrates three possible configuration of jobs $[i]$, $[i+1]$ and of A.

(i) If $A \leqslant C_{[i]} < C_{[i+1]}$, pairwise interchange will decrease the total penalty (Fig. 2 (*a*)).

(ii) If $C_{[i]} \leqslant A < C_{[i+1]}$, pairwise interchange will not affect the total penalty (Fig. 2 (*b*)).

(iii) If $C_{[i]} < C_{[i+1]} \leqslant A$, pairwise interchange will also not affect the total penalty (Fig. 2 (*c*)).

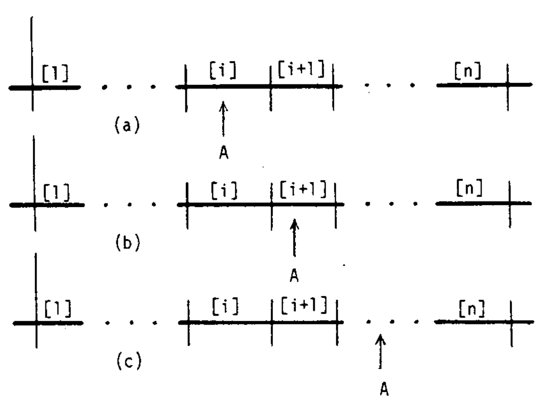

Figure 2. Three relative positions of A.

Optimality of SPT follows, since successive pairwise interchanges from any arbitrary sequence as indicated above will result in the SPT sequence without increasing total penalty.

Etant donné les temps de traitement de n travaux sur une unique machine avec des pénalités pour retard et avance et des pénalités associées à l'assignation de dates-limite, l'objectif est de sélectionner les dates-limite et la séquence optimales. La procédure de planification pour la résolution de ce problème est présentée avec des preuves d'optimalité et une illustration par exemples numériques.

Wenn man von den Ausführungszeiten für n Arbeitsabläufe auf einer einzelnen Maschine ausgeht, verbunden mit Strafen für zu frühe oder zu späte Erledigung und Zuteilung von Fälligkeitsdaten, bestecht das Ziel darin, optimale Fälligkeitsdaten und eine optimale Bearbeitungsfolge festzulegen. Es wird ein Planverfahren für die Lösung dieses Problems vorgelegt, und zwar mit Optimalitätsnachweis und veranschaulichenden numerischen Beispielen.

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