Technical Appendix to Fat Products

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1 Bounded Interval and Linear Costs Monopoly

I prove the Monopolist Theorem under linear development cost assumption and a bounded interval. Let the measure cost function $C([a,b]) = C_M \times (b-a)$, where C_M is a positive constant. The proof is a simple two step procedure, where in the first step the optimal p is found for a fixed $m = M$, and then I maximize with respect to m. The complications arise since the SOCs are not satisfied, and the profit function is a piecewise function – a convex parabola until a cutoff after which the function is linear.

Theorem 1 *(Fat Product Monopolist). A monopolist who has the ability to offer a Fat Product will offer a product of measure* 1 *for the price of* R *if and only if* $C_m \leq R - \dfrac{R^2}{2t}$. *Otherwise the monopolist will offer a standard point product for* $p^* = \frac{R}{2}$.

Proof. First I Calculate the optimal price for a given measure. The demand is $D(p) = M + \dfrac{2(R-p)}{t}$, and therefore the problem is the following (since there are no production costs, and the development costs are fixed):

$$\begin{gathered}\max p(M + \frac{2R}{t}) - \frac{2p^2}{t} \\ s.t. D(p) \leq 1 \\ p \leq R\end{gathered}$$

transforming into the standard form:

$$\begin{gathered}\max -\frac{2}{t} \times p^2 + (M + \tfrac{2R}{t}) \times p \\ \frac{2}{t} \times p + 1 - M - \frac{2R}{t} \geq 0 \\ -p + R \geq 0\end{gathered}$$

If there are any Kuhn-Tucker points they must satisfy the following conditions:

$$-\frac{4}{t} \times p + M + \frac{2R}{t} + \lambda_1 \frac{2}{t} - \lambda_2 = 0 \tag{1a}$$

$$(\frac{2}{t} \times p + 1 - M - \frac{2R}{t}) \times \lambda_1 = 0 \tag{1b}$$

$$(-p + R) \times \lambda_2 = 0 \tag{1c}$$

$$\lambda_1, \lambda_2 \geq 0 \tag{1d}$$

Consider what happens when $\lambda_1 = 0$ and $\lambda_2 = 0$. We get $p = \frac{mt+2R}{4}$, and $p \times D_+(p) = pm + \frac{2Rp - 2p^2}{t} = \frac{(mt+2R)^2}{8t}$.

Consider $\lambda_1 > 0$ and $\lambda_2 = 0$. From the second condition on the Kuhn-Tucker points we get $p = -\frac{t}{2} + \frac{Mt}{2} + R$, then from the first condition we get: $\lambda_1 = -t + \frac{Mt}{2} + R$. The revenue in this case will be simply p, since the demand is 1. Since $\lambda_1 > 0$, for this case to happen $-t + \frac{Mt}{2} + R > 0 \Longrightarrow M > 2 - \frac{2R}{t}$.

Consider $\lambda_2 > 0$ and $\lambda_1 = 0$. From the third condition on the Kuhn-Tucker points we get $p = R$. Then from the first condition we get $\lambda_2 = M - \frac{2R}{t}$, so for this case to happen we need $M > \frac{2R}{t}$. In this case the demand will just be M, and so the revenue will be RM.

Consider $\lambda_1 > 0$ and $\lambda_2 > 0$. From the third condition on the Kuhn-Tucker points we get $p = R$. Plugging this into the second condition we get $M = 1$. In this case the revenue is R.

Now that we have bounds on M for both case 2 and case 3, and since we know that the cases are mutually exclusive, combine the bounds, and get for case 2: $2 - \frac{2R}{t} < M < \frac{2R}{t}$ and for case 3: $\frac{2R}{t} < M < 2 - \frac{2R}{t}$. Therefore, if $2R > t$ then case 2 will happen, and case 3 can not happen, and the opposite if $2R < t$, and if $2R = t$, neither of them can happen. Also notice that as M goes to 1, both case 2 and case 3 solutions go to case 4, and if $2R = t$, then case 1 solution goes to case 4 as well. Therefore we do not need to worry about case 4 being a special case. Overall, in the next step we will have to worry about three scenarios, here is the summary:

Condition	$2R = t$	$2R > t$	$2R < t$
Start with	$p = \frac{mt + 2R}{4}$	$p = \frac{mt + 2R}{4}$	$p = \frac{mt + 2R}{4}$
Switch when	Don't switch	$M = 2 - \frac{2R}{t}$	$M = \frac{2R}{t}$
Switch to	...	$p = \frac{t}{2} + \frac{Mt}{2} + R$	$p = R$

Now we need to calculate the optimal measure given the price. I proceed by examining each of the cases above. Consider $2R = t$ first. Then, $\Pi_+(m) = \frac{(mt+2R)^2}{8t} - C_m \times m$, which is clearly a convex parabola in m. Therefore the minimum lies at one of the endpoints, therefore we just need to compare profits. $R - C_m \geq \frac{R^2}{2t}$ for $m = 1$ to be the maximum, so if $C_m \leq R - \frac{R^2}{2t} = \frac{3R}{4} = \frac{3t}{8}$ then it is optimal to have $m = 1$, and $p = R$. Profit is then $R - C_m$, and the consumer welfare is clearly zero. Otherwise the optimum is $m = 0$, and we get back to the Lemma 1 solution.

For $2R > t$ scenario the profit function is the same until the switch point, so let's assume that the optimal solution on that interval is the switch point (since the function is convex), and then compare the answers to $\frac{R^2}{2t}$. This gives us $\Pi_+(2 - \frac{2R}{t}) = \frac{(2t-2R+2R)^2}{8t} - C_m \times (2 - \frac{2R}{t})$. After the switching point the profit function becomes price − cost, so the switch point falls under this definition as well. The derivative of profit w.r.t. m becomes $\frac{t}{2} - C_m$. Therefore if $t \geq 2C_m$ then the optimal solution is $m = 1$, $p = R$. Let's compare it with $m = 0$ solution. Again, we get if $C_m \leq R - \frac{R^2}{2t}$, then $m = 1$ is optimal, and otherwise m = 0. If $t < 2C_m$ then $m = 2 - \frac{2R}{t}$, and $p = t/2$. Demand in this case will be 1, and so the profit will be $\frac{t}{2} - C_m \times (2 - \frac{2R}{t})$. Let's compare this with the profit from the $m = 0$ case. So for the measure to be positive, $\frac{t}{2} - C_m \times (2 - \frac{2R}{t}) \Longrightarrow C_m \times (2 - \frac{2R}{t}) \leq \frac{t^2 - R^2}{2t} \Longrightarrow C_m \leq \frac{t+R}{4}$. However, since $R < t < 2C_m$, we get $C_m \leq \frac{t+R}{4} < \frac{2t}{4} < C_m$, meaning that the first inequality never holds, so this sub case never occurs. Otherwise we go back to Lemma 1.

For the last, $2R < t$, scenario, the same thing happens as above before the switch point, and then after the switch point we get the profit function equal to $(RM - cost)$, with the switching point following this definition as well, so the derivative w.r.t. m becomes $R - C_m$. If $R \geq C_m$, then the optimal solution is $M = 1$, $p = R$, so the comparison with $m = 0$ is routine by now: we get if $C_m \leq R - \frac{R^2}{2t}$, then $m = 1$ is optimal, and otherwise $m = 0$. If $R < C_m$ then $M = \frac{2R}{t}$, $p = R$. Demand in this case will be M, and so the profit will be $\frac{2R(R - C_m)}{t}$, and the consumer welfare will be 0, since the price is the reservation price. Let's compare with the $m = 0$ profit. We get $\frac{2R(R-C_m)}{t} \geq \frac{R^2}{2t} \Longrightarrow 4R^2 - 4RC_m \geq R^2 \Longrightarrow C_m \leq \frac{3R}{4}$ for the measure to be positive. ■

2 Relaxing Linear Transportation Costs Assumption[1]

Linear transportation costs for the customers is an assumption made throughout the literature on spatial models. However it is not clear why should that assumption be close to reality even for simple applications of spatial models, where the transportation costs actually represent the physical costs of going from one place to another, let alone applications where the costs represent how far away is the product from the customer's ideal product in some characteristics space. Even with physical transportation costs, there is an area where the customer can just walk, then there might be an area covered by the public transportation system, and so on, and there is no reason for the costs to vary linearly within each of the areas.

It would be a major drawback of the model if I get the results because of the linear transportation costs. Therefore it would be natural to assume some transportation costs function $t(d)$, where d is the distance from the customer to the product offered by a firm and $t(\cdot)$ is a strictly increasing differentiable function, with $t(0) = 0$. For simplicity, assume that the firms are in the equilibrium where all the consumers are served and that the profit function is concave in the price and measure of the product.

Proposition 1 *With transportation costs a function $t(\cdot)$, in the symmetric equilibrium firms' charge price $p = \dfrac{t'(\frac{1}{2N} - \frac{m^*}{2})}{N}$ and make products of measure m^*, such that $c'(m^*) = \dfrac{p}{2}$.*

Proof. I examine two firms, one with base at 0, which will have a product of measure m and charge p for it, and the other one located at $\frac{1}{N}$, with the product of measure m^*, charging p^* for it - set up analogous to the one in the proof of Theorem 2. Consider a customer located at x between the two firms. Then the customer's utility from buying the two products are, respectively, $R - t(x - \frac{m}{2}) - p$ and $R - t(\frac{1}{N} - \frac{m^*}{2} - x) - p^*$. To find out the customer indifferent between the two products, just make the two utilities equal, and simplify to get

$$p - p^* = t(\frac{1}{N} - \frac{m^*}{2} - x^*) - t(x^* - \frac{m}{2}). \tag{2}$$

Assuming the other neighbor is also playing m^* and p^*, the demand for firm at 0 is $2x^*$. Therefore the profit of this firm will be $\Pi(p, m) = 2x^* \times p - c(m)$. Then, implicitly differentiating equation 2, with respect to p one can show that

$$\frac{\partial x^*}{\partial p} = \frac{-1}{t'(\frac{1}{N} - \frac{m^*}{2} - x^*) + t'(x^* - \frac{m}{2})}. \tag{3}$$

Then, implicitly differentiating equation 2, with respect to m, one can show that

$$\frac{\partial x^*}{\partial m} = \frac{t'(x^* - \frac{m}{2})}{2\left[t'(\frac{1}{N} - \frac{m^*}{2} - x^*) + t'(x^* - \frac{m}{2})\right]}. \tag{4}$$

Now we can take look at the first order conditions of the profit function:

$$\frac{\partial \Pi}{\partial p} = 2x^* + 2p \times \frac{\partial x^*}{\partial p} = 0, \tag{5a}$$

$$\frac{\partial \Pi}{\partial m} = 2p \times \frac{\partial x^*}{\partial m} - c'(m) = 0. \tag{5b}$$

Notice, that if we enforce the symmetry assumption to 2 ($p = p^*$ and $m = m^*$), we know that $t(\frac{1}{N} - \frac{m^*}{2} - x^*) = t(x^* - \frac{m}{2})$, and therefore $x^* = \frac{1}{2N}$. Substituting from 3 and 4 into 5a:

$$p = \frac{t'(\frac{1}{2N} - \frac{m}{2})}{N}. \tag{6}$$

[1] Thanks to Michael Whinston for raising this issue.

And from 5b I get:

$$c'(m) = \frac{p}{2}. \tag{7}$$

Again, if this m is bigger than $\frac{1}{N}$ than the firms end up playing the standard Bertrand, going down to marginal costs. However then the firms will have a profitable deviation to $m = 0$, and therefore there will be no symmetric equilibrium in this case. ■

I have derived the optimal price and measure, however the interesting results were what happens with the measure if the firms can restrict themselves, and what happens with the profits as the transportation costs go up.

Proposition 2 *It would be profitable for all the firms in the market to commit to making only* $m = 0$ *(standard) products if and only if* $\left[t'(\frac{1}{2N} - \frac{m^*}{2}) - t'(\frac{1}{2N})\right] < c(m^*)N^2$*. In particular, this is satisfied if the transportation cost function* $t(\cdot)$ *is convex.*

Proof. Similarly to the proof of the previous proposition, we can derive the optimal price if the firms are restricted to price at $m = 0$. This price turns out to be $\frac{t'(\frac{1}{2N})}{N}$. Therefore, it is profitable for the firms to sign a stand still agreement with respect to measure iff the profit with $m = 0$ is higher than the one from the previous proposition, or

$$\left[t'(\frac{1}{2N} - \frac{m^*}{2}) - t'(\frac{1}{2N})\right] < c(m^*)N^2. \tag{8}$$

Since the left hand side is bigger than zero, and $m^* > 0$, then if $t'(\cdot)$ is an increasing function, the right hand side is less than zero, and so the inequality is satisfied. ■

While this condition is not as clear as the one from corollary in the linear travel cost case, where this was always satisfied, there is still a wide range of values where this condition holds. As the transportation costs become more and more steep, the firms have to invest more into the cost of developing fat products, giving us the result. However, if the transportation costs are sufficiently concave, then an increase in the optimal measure takes the marginal customer lower along the transportation cost function and allows the firms to charge a higher price.

Since the transportation costs are now a function, to make comparative statics, I look at transportation costs $\alpha \times t(\cdot)$, at $\alpha = 1$, and see how does increasing α affect the firms' profits.

Proposition 3 *In equilibrium profits go down as the transportation costs go up (market becomes more differentiated) if and only if* $c''(m^*) < \frac{t'(\frac{1}{2N} - \frac{m^*}{2})}{4}$.

Proof. With the new term α, the condition for the optimal measure stays the same ($c'(m^*) = \frac{p}{2}$) and the one for the optimal price becomes

$$p^* = \frac{\alpha t'(\frac{1}{2N} - \frac{m^*}{2})}{N}. \tag{9}$$

Therefore the profit for each firm is now $\Pi(p^*, m^*) = \frac{\alpha t'(\frac{1}{2N} - \frac{m^*}{2})}{N^2} - c(m^*)$. Before we differentiate it with respect to α, we have to derive $\frac{\partial m^*}{\partial \alpha}$ first. We will do it implicitly from the combination of new optimal price and the optimal measure conditions, and simplifying get

$$\frac{\partial m^*}{\partial \alpha} = \frac{2t'(\frac{1}{2N} - \frac{m^*}{2})}{\alpha t''(\frac{1}{2N} - \frac{m^*}{2}) + 4Nc''(m^*)}. \tag{10}$$

Now we can differentiate the profit[2], and check when is the derivative less than zero (getting rid of common terms):

$$1 < \frac{\alpha t''(\frac{1}{2N} - \frac{m^*}{2}) + \alpha N t'(\frac{1}{2N} - \frac{m^*}{2})}{\alpha t''(\frac{1}{2N} - \frac{m^*}{2}) + 4Nc''(m^*)}. \tag{11}$$

A necessary condition for this inequality to hold is that the denominator has to be positive, since the numerator is always positive. Given that we have the denominator positive, and since $\alpha \to 1$:

$$c''(m^*) < \frac{t'(\frac{1}{2N} - \frac{m^*}{2})}{4}. \tag{12}$$

■

Again, we get the conclusion that it is possible to overall lossses as the firms become more differentiated. Moreover, the condition looks similar to the condition in the proposition from the linear travel cost case, which was $c''(m^*) < \frac{t}{4}$, and of course with linear costs t is the derivative. Overall, linear travel costs assumption was not necessary to achieve any of the results.

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[2] $$\frac{\partial \Pi}{\partial \alpha} = \frac{t'(\frac{1}{2N} - \frac{m^*}{2})}{N^2} - \frac{\alpha t''(\frac{1}{2N} - \frac{m^*}{2})}{2N^2}\frac{\partial m^*}{\partial \alpha} - c'(m^*)\frac{\partial m^*}{\partial \alpha} =$$
$$= \frac{t'(\frac{1}{2N} - \frac{m^*}{2})}{N^2} - \frac{\alpha t'(\frac{1}{2N} - \frac{m^*}{2})t''(\frac{1}{2N} - \frac{m^*}{2})}{N^2 \times \left[\alpha t''(\frac{1}{2N} - \frac{m^*}{2}) + 4Nc''(m^*)\right]} - \frac{\alpha N \left[t'(\frac{1}{2N} - \frac{m^*}{2})\right]^2}{N^2 \times \left[\alpha t''(\frac{1}{2N} - \frac{m^*}{2}) + 4Nc''(m^*)\right]}.$$