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Analyzing the Simultaneous Use of Auctions and Posted Prices for Online Selling

Hila Etzion
Stephen M. Ross School of Business, University of Michigan, Ann Arbor, Michigan 48109, etzionh@bus.umich.edu

Edieal Pinker, Abraham Seidmann
W. E. Simon School of Business Administration, University of Rochester, Rochester, New York 14618
{pinker@simon.rochester.edu, seidmannav@simon.rochester.edu}

Many firms in the business-to-consumer market sell identical products online using auctions and posted prices at the same time. In this paper, we develop and analyze a model of the key trade-offs sellers face in such a dual-channel setting built around the optimal choice of three design parameters: the posted price, the auction lot size, and the auction duration. Our results show how a monopolist seller can increase his revenues by offering auctions and a fixed price concurrently, and we identify when either a posted price only or a dual-channel strategy is optimal for the seller. We model consumer choice of channels, and thus market segmentation, and find a unique (symmetric) auction-participation equilibrium exists in which consumers who value the item for more than its posted price use a threshold policy to choose between the two channels. The threshold defines an upper bound on the remaining time of the auction. We explain how optimizing the design parameters enables the seller to segment the market so that the two channels reinforce each other and cannibalization is mitigated. Our findings also demonstrate that there are two dominant auction design strategies in this setting: one-unit auctions that tend to be short and long multiunit auctions. The optimal strategy for the seller depends on the consumer arrival rate and the disutility of delivery delay incurred by high-valuation consumers. In either case, the optimal design of the dual channel can significantly outperform a single posted-price channel. We show even greater benefits over a naive approach to managing the two channels that optimizes each independently. Our results suggest that unless firms jointly manage these online channels, they may find that adding auctions actually reduces their revenues.

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1. Introduction

In the business-to-consumer market, many firms are selling the same or almost identical products online using auctions and fixed prices simultaneously. The Internet enables firms to operate the two venues in the same space (the World Wide Web) and time and allows consumers to observe and compare the two selling channels with no additional costs. For example, IBM offers selected products via auctions on eBay while the same products are sold for posted prices on IBM's website; CompUSA conducts auctions for new and refurbished products on a dedicated auction website, while selling identical items for posted prices on its catalog website; Sam's Club operates the two selling channels within the same website; airline tickets are sold for fixed prices as well as via reverse and forward auctions.

The practice of operating auctions and fixed prices in parallel, on the Internet, raises many important questions. Clearly, the two selling channels cannot be treated independently. Optimizing each channel separately results in a suboptimal global design, as the two channels compete in the same market. Although the problem of optimally selecting and designing a single selling mechanism (auction or posted price) has been well addressed in the literature, there is little research on how to operate and design such selling mechanisms in parallel. In addition, the economic benefits and limitations of using both auctions and posted prices are not clear. On one hand, an auction creates a venue for selling to consumers who do not value a product as much as the posted price, thereby increasing revenues. On the other hand, the auction channel may attract consumers who otherwise would have bought at the posted price, thereby reducing revenues. Our research focuses on the following issues:

- How do consumers behave when faced with the choice between the two channels?
- What is the optimal choice of the auction parameters, i.e., auction length and quantity, when identical items are sold for a posted price by the same outlet?
- What is the optimal posted price when identical items are being auctioned?
- When does the dual channel outperform the single channel (posted price only)?

We develop a mathematical model that addresses these questions. In our model, a monopolist sells identical items using a sequence of sealed-bid second-price auctions and a posted price at the same time. Our research contributes to the existing auction literature by introducing the following three changes. First, the auctions are conducted parallel to a posted price and serve the same stream of consumers, so arriving consumers can choose between purchasing the item and bidding. Second, in our model the number of bidders is stochastic, and consumers can arrive at any time during the auction. Thus, different consumers spend different amounts of time in the auction and incur different delay costs, depending upon their arrival time. Finally, the auctioned quantity is an endogenous decision variable, set by the seller to maximize total (auction and fixed-price) revenue.

Our results rely on a model of consumer behavior that defines how consumers choose between bidding and purchasing the item for its posted price, when the seller has an unlimited supply of the item. We prove that there exists a symmetric equilibrium in which consumers who value the item for more than its posted price, the "high-valuation consumers," use a threshold policy to choose between the two selling channels. The threshold defines an upper bound on the remaining time of the auction: if the remaining time observed by a high-valuation consumer upon his arrival exceeds the threshold, the consumer chooses to purchase the item for its posted price. If the remaining time is less than the threshold, the consumer chooses to participate in the auction. We also show that the optimal bidding strategy in the sealed-bid second-price auction is no longer truth-revealing. For a consumer who values the item for more than the posted price, bidding his true valuation is weakly dominated by placing a bid equal to the posted price.

We formulate a nonlinear optimization problem for choosing the posted price, auction quantity, and auction length when the seller's objective is to maximize expected revenue per unit time. Based on many numerical experiments, we argue that the optimal posted price in the dual channel is unique and is higher than the optimal posted price in the absence of auctions, and that a seller can significantly increase his revenue by adding an auction channel to the posted-price channel. Depending on the model's parameter values, we find that optimization results in one of the following two strategies: (1) the seller should conduct one-unit auctions and decrease the auction length as the consumer arrival rate increases, or (2) the seller should conduct long auctions and increase the size of the auction lot as the consumer arrival rate increases. In the first strategy, the number of units auctioned per consumer is small; in the second, this ratio is significantly higher.

This result reflects the main trade-offs a seller faces: Setting one-unit auctions or long auctions deters high-valuation consumers from bidding, but these settings also reduce the number of units sold via auction per unit time, reducing total sales. When the seller offers one-unit lots, he should make the in auctions short enough to have a significant increase in sales while maintaining a high auction price. Thus, the auction length is decreasing with the consumer arrival rate. When the seller uses long auctions, he should offer larger lots to increase sales, but since increasing the quantity offered also decreases the auction price, the auction lot is increasing in the arrival rate. Which of these two strategies is optimal for the seller depends on the consumer arrival rate and the delay cost per unit time incurred by high-valuation consumers. Our results confirm that the seller's revenue from the dual channel can be higher than the optimal revenue achieved by using a posted price alone, or by managing the two channels independently.

The paper's structure is as follows. In §2, we review the relevant literature. In §3, we consider a monopolist seller and construct a model of selling identical items using auctions and a posted price simultaneously. We develop a detailed model of consumer behavior and show how to calculate the seller's expected revenue from the dual channel for a stochastic number of bidders. We describe the characteristics of an optimal design of the dual channel through numerical examples in §4, and we conclude in §5.

2. Literature Review

Auction markets have been of interest to researchers for generations. An extensive literature exists that examines the optimal design of auctions and the ranking of different auction mechanisms: McAfee and McMillan (1987a), Milgrom (1987), and Klemperer (1999) provide excellent surveys. Traditionally, analysis of auction design has focused on the auction mechanism itself. In these analyses, an auction is fully characterized by how bidder valuations are revealed and how the actual goods are allocated. Much focus has been on showing under which circumstances common auction mechanisms are equivalent and on proving when these mechanisms are truth revealing. The widespread use of online auctions has brought to the forefront a new set of managerial problems. Pinker et al. (2003) survey the current state of research on the specific problems faced in the design of such auctions.

The problem of optimally selecting and designing a single selling mechanism has been well addressed. Wang (1993) considers the impact of the dispersion in the distribution of buyers' valuations on the choice between a posted price and an auction for a seller selling one unit of a good. He demonstrates that an auction is preferred when buyer valuations become more disperse. His model, like many others, ignores buyers' costs associated with auctions, such as the cost of delays and the cost of monitoring the auction. Wang does not model how consumers would choose between the two mechanisms, because he does not consider simultaneous use of both. Hence, the set of consumers is identical for both methods (it does not depend on the choice of mechanism). Harstad (1990) uses a model in which the seller's choice of selling method and reserve price does affect the number of bidders attending the auction. Ehrman and Peters (1994) consider a waiting cost for bidders due to the disappearance of outside alternatives. De Vany (1987) considers a seller with one unit of a commodity choosing between three mechanisms: an auction after a fixed time T, an auction after a fixed number of consumers have arrived, and posted price. Consumers incur a cost of waiting when an auction mechanism is chosen.

Some researchers have considered the choice of mechanism when the seller offers multiple homogenous units of the good (Harris and Raviv 1981, Riley and Zeckhauser 1983, Maskin and Riley 1989, Arnold and Lippman 1995). Others have tried to explain the coexistence of the different selling mechanisms in a market and have examined the equilibrium of mechanisms in a competitive environment (Kultti 1997, Epstein and Peters 1999, Peters 1999). Recently, Gallien (2002) has compared the fixed price, dynamic posted price, and online auction mechanisms when selling multiple units to risk-neutral and time-sensitive consumers. Each buyer is characterized by his valuation of the item and his arrival time, and a buyer's net value decreases in the interval between his arrival and the time he obtains the item. Yet, although mechanism selection has been well covered, there is little research on how to operate and design such selling mechanisms in parallel. In addition, the economic benefits and limitations for a firm that concurrently employs multiple selling mechanisms are not clear.

Vakrat and Seidmann (1999) study simultaneous sales of identical products using online auctions and a fixed-price catalog. Their empirical research finds that the auctions result in an average discount of 25% relative to the catalog prices. They model a one-unit English auction and assume that the number of bidders is deterministic, and that consumers have full information about the catalog price. They find that the expected auction price is a function of the number of bidders and of delay and search costs associated with the auction. Their paper does not model consumers' choice between the two channels, nor does it show what incentives the seller has to conduct such an auction. Van Ryzin and Vulcano (2004) examine the optimal pricing-replenishment policy when the firm sells in two markets—one fixed-price market and one auction market—and demand comes from two different and independent streams of customers, so there is no need to model consumers' choice. In their model, the seller decides how to split the inventory between the two markets.

Another use of the two selling mechanisms simultaneously is the "buy-now" price offered on many C2C auction sites. On Yahoo Auctions, for example, the auction will automatically close when a bidder meets the specified buy-now price, and the item is sold to that bidder. Another example is eBay's "Buy It Now" option, which is only shown on listings until an item receives its first bid, or, when the seller sets a reserve price, until the reserve is met. In this business model, the auction is the main selling venue, and the buy-now price is the secondary channel. Budish and Takeyama (2001) model an English auction with a buy-now price. Their model has only two bidders and two possible types: a high-valuation consumer and a low-valuation consumer. They show that the seller is strictly better off by adding a buy-now price to the auction only when bidders are risk averse. This result, however, does not hold in a more general framework with N valuations. Hidvegi et al. (2002) model an auction with N bidders having continuously distributed private valuations and show that a bidder with a very high valuation compared to the buy price will use the buy price unconditionally, a bidder with a valuation close to the buy price will only use the buy price when the current bid reaches a threshold price, and there is no change in the optimal bidding strategy for a bidder with a valuation lower than the buy price. They find that when either party is risk-averse, a buy-price auction is strictly better for the seller. They do not consider delay costs for bidders. Reynolds and Wooders (2003) show that, when bidders are risk-neutral, an auction with a buy-now price is revenue equivalent to the standard English ascending-bid auction, so long as the buy-now price is not too low. When bidders are risk-averse, however, auctions with buy-now prices are advantageous for the seller. They compare the seller's revenue and the bidders' payoff for two auctions formats: the Yahoo format and the eBay format.

It is important to mention that, with the exception of Gallien (2002), the existing research does not model the facts that different consumers arrive at different stages of the online auction and that their expected utility from bidding is a function of their arrival time. Even those papers that associate a delay cost with bidding assume that all bidders spend the same amount of time in the auction and thus incur the same delay cost. Clearly, such assumptions are not suitable for the modeling of online auctions, which can last as long as a week or more. Our model addresses this issue.

3. The Model

We model an online seller who offers identical items using two selling mechanisms, posted price and auctions, simultaneously. The auctions have a fixed duration and are then repeated. The seller's objective is to maximize his revenue per unit time. The seller chooses the auction duration T, the quantity to auction q, and the posted price p. Without loss of generality, we assume that the marginal cost of each unit is zero (if this is not so, consumers' valuations of the product can be taken net of the marginal cost). The seller's publicly declared reserve price is R.[1] We also assume that the seller can satisfy any demand. Consumers visit the website according to a Poisson process with rate λ, and each consumer is interested in purchasing one unit of the good.[2] Consumers have independent private values for the good. We assume that each consumer's valuation, V, is independently drawn from a probability distribution with cumulative density function $F()$ with support set $[\underline{v}, \overline{v}]$, where $\underline{v} \geq R$.

Because the two selling channels are offered simultaneously on the same platform or in the same space (the Internet), we assume that consumers can observe both channels on arrival, with no additional costs. Hence, consumers are fully informed: they observe the item's posted price, whether an auction is currently offered, the auctioned quantity, and the time remaining in the auction. They do not know the number of other bidders.

We model the auctions using the sealed-bid $(q+1)$-*price* format with risk-neutral bidders having unit demand and independent private values for the

[1] A public reserve price is equivalent to setting a minimum initial bid.

[2] We assume a constant arrival rate. Notice that in B2C auctions, although there is a lot of last-minute activity because of "sniping," there is a distinction between arrival times and activity/bidding times.

good. In a sealed-bid $(q+1)$-*price* auction, the winners are the bidders with the q highest bids (q being the auctioned quantity), and each pays a price equal to the $(q+1)$ highest bid (the highest losing bid). In such a sealed-bid $(q+1)$-*price* auction, with no posted price offering, the dominant strategy for each bidder is to bid his true valuation of the item (Milgrom 1987). By doing so, the bidder is setting an upper bound on the price that he is willing to pay: he will accept any price below his reservation price and none above.

Most online auctions are conducted using the open, ascending-bid English auction format—bidders can observe the lowest bid needed to win at every moment of the auction and, on some websites, can see how many other bidders there are. This suggests that consumers have more information available to them when choosing between the posted price and auction participation than in the sealed-bid auction we model. In practice, however, last-minute bidding or "sniping" is very prevalent (Roth and Ockenfels 2002). This suggests that actually very little information is available to the consumers—they do not know how many other bidders are lurking in the background, nor do they have any indication of what these other bidders are willing to bid. The result is a de facto sealed-bid auction.

In our setting, the existence of the posted-price option adds considerable complexity to the analysis of the auction because it splits the consumers into several subgroups (see Figure 1).

The posted price first splits the consumers into those with low valuations (i.e., valuations less than the posted price), and those with high valuations (i.e., valuations greater than or equal to the posted price). In this paper, all low-valuation consumers become bidders in the auctions; we explain the rationale for this in §3.1. A fraction β of the high-valuation consumers also become bidders, while a fraction $1-\beta$ purchase at the posted price. The probability of participation, β, depends on the design variables (q, T, and p), and much of the analysis in this section is devoted to their determination. Some bidders win the auction and some lose. The high-valuation losing bidders will purchase at the posted price at the auction's conclusion. To explain the choices made by consumers depicted in Figure 1, we next describe our model of consumer behavior. Table 1 summarizes the notation used throughout the paper.

3.1. Consumer Behavior

As noted in the literature review, most auction models examine markets in which auctions are the sole-selling mechanism and the number of bidders is deterministic. In such markets, consumers face a simple choice between bidding and not. In the absence of auction-related costs, the expected value from bidding is always nonnegative, so the set of bidders is the same as the set of arrivals: each arriving consumer chooses to participate in the auction rather than stay out of the market. Here, we model the behavior of risk-neutral consumers when the seller offers both auctions and a posted price. We examine how consumers choose between the available channels and define a weakly dominant bidding strategy (and thus a dominant equilibrium) for those who choose to bid.

When a risk-neutral consumer has the option of a posted-price channel, he bases his choice on his expectation of a greater surplus. When consumers

Figure 1 **Schematic of Customer Splitting in the Presence of Dual Channel**

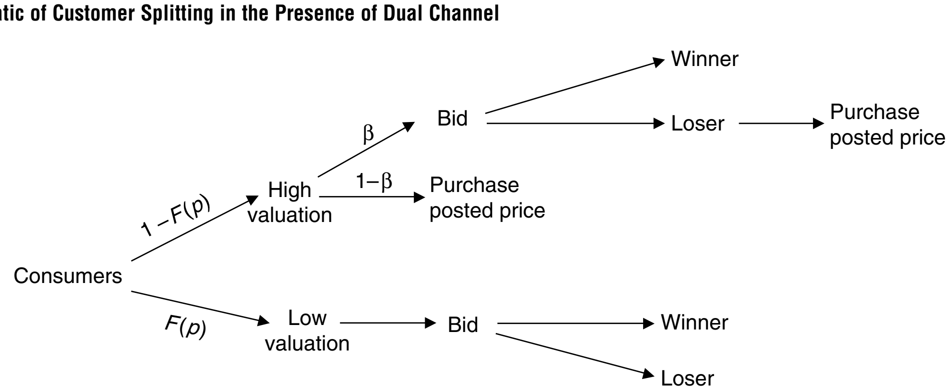


Table 1 Summary of Notation

Decision variables:	
p	posted price.
q	auctioned quantity (per auction).
T	auction length.
Model parameters:	
λ	arrival rate of consumers at the website.
$F(v)$	cumulative density function of consumers' valuation distribution with support $[\underline{v}, \overline{v}]$.
w	delay cost incurred by high-valuation consumers per unit time.
R	seller's public reserve price.
Other notation:	
p_a	auction closing price.
N^+	random number of bidders who value the item for more than its posted price.
N^-	random number of bidders who value the item for less than its posted price.
$\bar{t}$	time remaining in an auction beyond which high-valuation consumers will not participate for the (symmetric) participation-strategy equilibrium.
$O\{x, y\}$	expected value of the xth order statistic of y draws from the consumer valuation distribution truncated on $[\underline{v}, p]$.
β	probability a high-valuation consumer participates in the auction.

choose the auction channel, it is because they believe an opportunity exists to purchase the good at a discount over the posted price. Yet there are costs to participating in an auction, so in expectation, the auction price discount must exceed these costs. There are essentially two auction participation costs: the cost of monitoring and making bids and the cost of deferring the purchase of the good until the end of the auction. There is empirical evidence that these costs influence the behavior of auction participants.

Hann and Terwiesch (2003) study bidder behavior on a German name-your-price service that allows bidders to update their bids after only a few minutes. They find that bidders do not bid very frequently with small increments, as one would expect, but rather bid only a few times (typically less than four) with significant bid increments. They explain this behavior as demonstrating that there is a significant participation cost in these online negotiation settings. Lucking-Reiley (1999) conducts experiments with Internet auctions of collectible trading cards, comparing the profit generation of Dutch auctions with that of first-price auctions, which theory predicts to be equivalent. He finds, to the contrary, that Dutch auctions had closing prices on average 30% higher than first-price auctions. He speculates that one possible reason is that Dutch auctions were much longer and bidders might have been impatient to complete their purchase. Motivated by Lucking-Reiley's observation, Carare and Rothkopf (2001) develop decision and game theoretic models of slow Dutch auctions to show how including auction transaction costs related to the auction duration alter their outcomes. In their models, the value the bidder receives from the auctioned good decreases with the time spent in the auction; i.e., there is a delay cost. We use a similar approach in our model.

Some auction mechanisms require more active participation by the bidders and therefore have higher participation costs. The sealed-bid auction we model does not require the bidders to constantly monitor the auction's progress or analyze the behavior of other bidders. In our setting, therefore, it is reasonable to assume that the most significant component of the auction participation cost is the delay cost.

To develop the intuition behind our model, we consider the following numerical example.

Example 1. A seller is offering two computer keyboards in an auction while simultaneously selling them online for \$100. Let us say seven consumers $\{B_1, B_2, B_3, B_4, B_5, B_6, B_7\}$ with respective valuations {5, 10, 80, 90, 101, 110, 120} arrive at the website. Consumers B_1, B_2, B_3, and B_4 have no other option but to bid in the auction because the posted price is too high for them. We demonstrate in Lemma 1 that their optimal bidding strategy is to bid their valuations. If only these four consumers bid, the closing price of the auction will be \$10, with B_3 and B_4 winning. What about B_5? He can purchase the keyboard at the posted price and have it shipped to him immediately while receiving a surplus of \$1, but he might do better by bidding in the auction. We demonstrate in Lemma 1 that B_5's optimal bidding strategy is to bid the posted price \$100. If he were to bid this way against consumers B_1, B_2, B_3, and B_4, he would win and pay \$80, yielding a surplus of \$21. If B_5 arrived at the auction three days before it ended, choosing to participate would force him to incur a waiting cost because if he wins, he will receive the item three days later than if he had purchased at the posted price. If this cost were \$3 per day, his surplus would be reduced to \$12. If consumer B_6 also bid in the auction, the closing price would increase to \$90, reducing B_5's surplus further to just \$2. We can see from this example that a high-valuation consumer may find it worthwhile to participate in the auction if he anticipates receiving a discount over the posted price larger than the delay cost, where the discount is determined by the number and types of the other bidders.

3.1.1. The Consumer's Problem. We characterize each consumer by his valuation V, and the time remaining in the auction when he arrives, t^e. Low-valuation consumers, those with $V < p$, cannot buy the item for its posted price because the value they get from doing so is negative, so they choose between bidding and staying out of the market. These consumers prefer to receive the item earlier rather than later, and they may choose not to bid if the remaining time of the auction is significantly long (hence, we later set an upper bound on the feasible auction length when solving the seller's optimization problem in §4). However, because they have no other option for obtaining the item, we assume that the delay cost per unit time perceived by these consumers is significantly lower than the delay cost per unit time perceived by consumers who can obtain the item instantly by paying the posted price. To simplify the problem, we therefore assume that the delay cost per unit time is $w = 0$ for consumers with $V < p$. The optimization problem faced by these consumers is

$$\max\{U_-^A(V, t^e), 0\}, \quad \text{where}$$
$$U_-^A(V, t^e) = \max_{b \in [0, \infty)} \{\Pr(win \mid b)V - E[auction\_payment \mid b]\}. \tag{1}$$

We define $U_-^A(V, t^e)$ as the maximum expected value from participating in the auction for a consumer of type (V, t^e), when $V < p$, where the expected value is taken over the bids of all other bidders in the auction. $\Pr(win \mid b)$ is the probability that the consumer wins the item in the auction by bidding b, and $E[auction\_payment \mid b]$ is the expected auction payment by a bidder who bids b. Notice that $E[auction\_payment \mid b]$ differs from the expected auction price when bidding b, $E[p_a \mid b]$, because when the consumer loses in the auction, his expected auction payment is zero, but the auction price is not.

High-valuation consumers, those with $V \geq p$, would buy the item for its posted price if auctions were not offered. High-valuation consumers choose between buying the item for its posted price and participating in the auction. It is never optimal for these consumers to do nothing, because their utility from buying the item for the posted price is nonnegative. We assume that when high-valuation consumers purchase the item for its posted price, they obtain the item instantly. When they choose to bid, they are choosing to experience a delay in obtaining and using the item, because they must wait until the end of the auction. Hence, when choosing to bid, these consumers incur a delay cost that is an increasing function of the time remaining until the end of the auction. $U_+^A(V, t^e)$ denotes the maximum expected value from participating in the auction for a consumer with valuation $V \geq p$, and we define $U_+^B(V, t^e)$ as the value a consumer of type (V, t^e) derives from purchasing the item for the fixed price. A high-valuation consumer arriving with t^e time units remaining in the auction solves the following optimization problem:

$$\max_{i \in A, B} U_+^i(V, t^e),$$

where

$$U_+^A(V, t^e) = \max_{b \in [0, \infty)} \{\Pr(win \mid b)V - E[auction\_payment \mid b] + \Pr(lose \mid b)(V - p) - wt^e\} \tag{2}$$
$$U_+^B(V, t^e) = V - p.$$

$\Pr(win \mid b)$ is the probability that the consumer wins the item in the auction by bidding b, and $\Pr(lose \mid b) = 1 - \Pr(win \mid b)$. The consumer evaluates the expected payoff from bidding, using an optimal bidding strategy, and compares it with the payoff from purchasing the item for the posted price. The first two terms of the RHS of (2) give the expected value from bidding b when the product is not offered for a posted price. The existence of a posted price offering has two opposing effects on the auction's value for a high-valuation consumer:

- $\Pr(lose \mid b)(V - p)$: If the customer loses the auction, we assume he can and will purchase the item for the same posted price with a payoff of $(V - p)$. Hence, the existence of a posted price increases the expected value from participating in the auction by reducing the cost of losing the auction. We acknowledge that this assumption might not capture the bidder's optimal behavior, and so we might be underestimating the high-valuation consumer's payoff when choosing to bid. However, we do not believe that this assumption will be restrictive in practice. A high-valuation consumer who has entered one auction and lost has faced a lot of competition from other high-valuation consumers, and thus has no reason to expect a different outcome in a future auction; hence, he will be less likely to bid again.

- $-wt^e$: Because the consumer could have bought the product for the posted price and obtained the item instantly, he incurs a delay cost when he chooses to bid and wait until the end of the auction to receive the item. We assume that this delay cost is linear in the time remaining until the auction ends. It may be that w, the delay cost per unit time, is an increasing function of V. That is, a consumer who values the item more also relates a higher cost to a delay in using the item. To simplify the following analysis, we assume that w is positive and independent of V for consumers with $V \geq p$.

3.1.2. Optimal Auction Participation and Bidding. In determining the consumers' participation and bidding strategies, we restrict our attention to symmetric equilibria, in which all consumers adopt the same strategy. This is reasonable because consumers are symmetric in the sense that their valuations and arrival times are drawn from the same distributions. We determine the strategy of a high-valuation consumer of type (V, t^e) in two steps. In Lemma 1, we first derive a weakly dominant bidding strategy for all consumers who have chosen to participate in the auction. In Proposition 1, we then use this strategy as an input in determining a unique symmetric equilibrium for a high-valuation consumer participation strategy.

We model the sealed-bid auction as a static game of incomplete information. Conditioned on having chosen to participate in the auction, the arrival time of the consumer is irrelevant to the bidding strategy because the delay cost is sunk. Therefore, for bidder i, the consumer type space is $T_i = [\underline{v}, \bar{v}]$, the support of the consumer valuation distribution, and the action space is the space of possible bids, $B_i = [0, \infty)$. A strategy, $b(V)$, is a mapping from the type space to the action space. Because valuations are independent, player i believes that V_k for every $k \neq i$ is drawn from the CDF $F()$.

Next, we find a bidding strategy, $b_i(V_i)$, such that for any given number of bidders and combination of other bidders' actions $b_{-i} = (b_1, b_2, \ldots, b_{i-1}, b_{i+1}, \ldots, b_{N^- + N^-})$, and for any other bidding strategy, $b'_i(V_i)$, $U_i(b_i(V_i), b_{-i}, V_i) \geq U_i(b'_i(V_i), b_{-i}, V_i)$ with strict inequality for some b_{-i}. That is, we find a weakly dominant strategy for this static game of incomplete information, and such a strategy provides a dominant equilibrium.

Lemma 1. *A weakly dominant bidding strategy for risk-neutral bidders with independent private values in a sealed-bid $(q+1)$-price auction that is conducted parallel to a posted price, p, is the following*:

$$b(V) = \begin{cases} V & \text{for } V < p \\ p & \text{for } V \geq p. \end{cases} \tag{3}$$

The proof is in Appendix 1.

Note that the existence of an outside option at price p puts an upper bound of p on the bids placed by high-valuation consumers. This is in contrast to the optimal bidding strategy in a traditional sealed-bid second-price auction, in which it is optimal to bid one's valuation. Losing the auction in our setting is less of a loss because of the infinite posted-price supply.

For consumers with $V < p$, the value from participating in the auction is nonnegative, so all of these consumers choose to bid rather than stay out of the market. Hence, the number of these participants in the auction is the same as the number of arrivals. The number of low-valuation participants, N^-, is a random variable from a Poisson distribution with rate $\lambda_1 = \lambda F(p)$. Consumers with $V \geq p$ choose to participate in the auction rather than to buy the item for its posted price if and only if

$$\Pr(win \mid p)V - E[auction\_payment \mid p] + \Pr(lose \mid p)(V - p) - wt^e \geq V - p. \tag{4}$$

Proposition 1. *In a dual channel with a sealed-bid $(q+1)$-price auction in which bidders follow the strategy of Lemma* 1, *high-valuation consumers participate in the auction iff*

$$D \equiv \Pr(win \mid p)p - E[auction\_payment \mid p] \geq wt^e, \tag{5}$$

and there exists a unique (symmetric) equilibrium in which all high-valuation consumers choose to bid if and only if $t^e \le \bar{t}$, where $0 < \bar{t} < T$ and $\bar{t}$ is given by the solution of the fixed point equation

$$D(\bar{t}) = w\bar{t}, \tag{6}$$

if $D(T) < wT$, and $\bar{t} = T$ if $D(T) \ge wT$.

The proof is in Appendix 1.

We note that $\Pr(win \mid p)p - E[auction\_payment \mid p]$ is the expected discount a high-valuation consumer gets over the posted price if he participates in an auction. We use $D(\bar{t})$ to denote the expected discount a high-valuation consumer gets over the posted price when all other high-valuation consumers use the threshold $\bar{t}$ to choose whether to participate in the auction or to buy the item for the posted price.

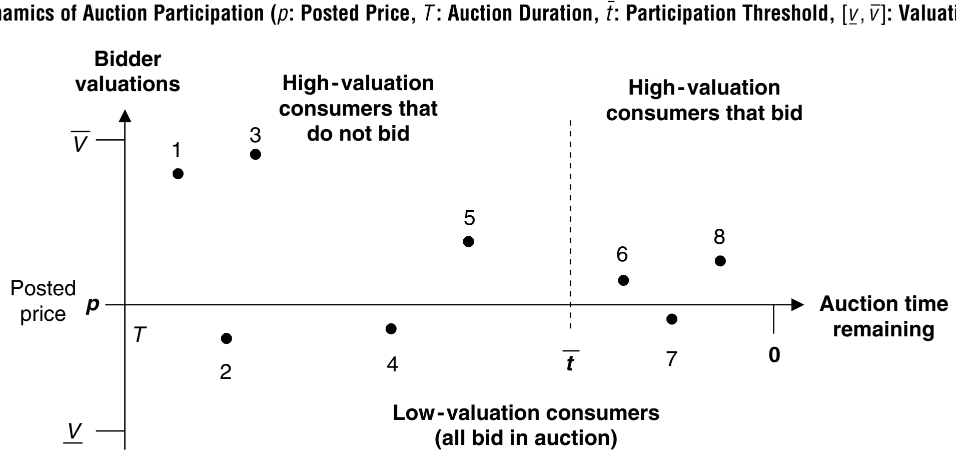
Based on Proposition 1, we conclude that under fairly general conditions high-valuation consumers use a threshold policy to choose between buying the item for its posted price and bidding in the auction. If the remaining time of the auction observed by a high-valuation consumer upon his arrival exceeds the threshold, the consumer chooses to purchase the item for its posted price. If the remaining time of the auction is less than the threshold, $\bar{t}$, the consumer chooses to participate in the auction. This result is shown in Figure 2. The numbered dots in Figure 2 depict the arrival times and valuations of bidders. The horizontal axis represents the time remaining in the auction when the consumer arrives, and the vertical axis represents the consumer's valuation of the good being sold.

Figure 2 depicts the two types of consumer segmentation occurring in the dual channel. The posted price splits the consumers into low- and high-valuation groups. The low-valuation consumers all bid in the auction regardless of their arrival times (Consumers 2, 4, and 7). The threshold time $\bar{t}$ divides the high-valuation consumers into those who bid and those who do not. Consumers 1, 3, and 5 all arrive with more than $\bar{t}$ time units remaining in the auction and therefore have delay costs that exceed their expected discount from participating in the auction. On the other hand, Consumers 6 and 8 arrive late enough that it is worthwhile for them to bid in the auction and delay their purchase of the item.

Because Poisson arrivals are uniformly distributed over a fixed-time interval, the fraction of high-valuation consumers that participates in the auction is given by $\beta = \bar{t}/T$, where $\bar{t}/T$ is the probability that a consumer arrives in the last $\bar{t}$ units of time of the auction. We conclude that the number of high-valuation consumers who bid, N^+, has a Poisson distribution with rate $\lambda_2 = \lambda(\bar{t}/T)(1 - F(p))$ and that the smaller the value of $\bar{t}$, relative to T, the more effectively the seller has segmented the low- and high-valuation consumers (i.e., fewer high-valuation consumers are expected to participate in the auction).

It is important to note that the results in Proposition 1 depend on the consumers' assumption that the seller's capacity is unlimited. When the capacity is limited, the probability of being able to purchase the item for the posted price at the end of the auction is less than one because of the positive probability that the seller will run out of stock. The

Figure 2 The Dynamics of Auction Participation (p: Posted Price, T: Auction Duration, $\bar{t}$: Participation Threshold, $[\underline{v}, \bar{v}]$: Valuation Support)



participation threshold would then depend on the consumer's valuation, V. Hence, our suggested model holds when consumers believe the probability that the seller will run out of stock during the auction is zero. This is plausible when the auction's length is relatively short and the seller's capacity is assumed to be large. A different model is needed for items such as airline tickets, end-of-season items, or refurbished goods, for which the probability of being out of stock is significant.

3.2. The Seller's Optimization

Recall that N^- and N^+ are defined, respectively, as the number of low- and high-valuation consumers who bid. We have shown above that N^- and N^+ are random variables from Poisson distributions with rates λ_1 and λ_2, respectively, where $\lambda_1 = \lambda F(p)$ and $\lambda_2 = \lambda(1 - F(p))\bar{t}/T$. The seller determines these rates by selecting T, q, and p. The seller's decision problem can thus be formulated as follows:

$$\max_{T,q,p} \frac{1}{T}\left[E[\pi_a] + (\lambda T(1-F(p)) - E[N^+])p + p\sum_{x=q+1}^{\infty}(x-q)\frac{e^{-\lambda_2 T}(\lambda_2 T)^x}{x!}\right], \tag{7}$$

where

$$E[\pi_a] = qp\sum_{x=q+1}^{\infty}\frac{e^{-\lambda_2 T}(\lambda_2 T)^x}{x!} + q\sum_{x=0}^{q}\sum_{y=q+1-x}^{\infty}O\{q-x+1,y\}\cdot\frac{e^{-\lambda_1 T}(\lambda_1 T)^y}{y!}\frac{e^{-\lambda_2 T}(\lambda_2 T)^x}{x!} + R\sum_{x=0}^{q}\sum_{y=0}^{q-x}(x+y)\frac{e^{-(\lambda_2+\lambda_1)T}(\lambda_2 T)^x(\lambda_1 T)^y}{x!y!}.$$

The first term of Equation (7) is the expected revenue from the auction. The second term is the expected revenue from sales for the posted price during the auction. The number of purchasers for the posted price equals the number of high-valuation consumers (consumers with $V \geq p$) who arrive during the auction less those who choose to bid. The last term in Equation (7) is the expected revenue from sales to high-valuation consumers who lost in the auction.

If the number of bidders is less than q, the auction price is the seller's reserve price R. Note that, in our model, R is a parameter—not a decision variable—and only consumers with valuations $V \geq R$ are relevant to the analysis. Our model could be used as an engine to identify the optimal reserve price in situations in which the reserve price is public knowledge. Some online auctions such as those conducted on eBay, SamsClub.com, and Compusaauctions.com, allow sellers to post secret reserve prices. A secret reserve price will deter some bidders from participating in the auction as Bajari and Hortacsu (2003) observe. Our model is not designed to capture this effect.

4. Design of the Dual Channel

To develop our intuition of the seller's perspective, consider a numerical example modeled on Example 1.

Example 2. Seven consumers $\{B_1, B_2, B_3, B_4, B_5, B_6, B_7\}$ with respective valuations $\{5, 10, 80, 90, 101, 110, 120\}$ come to a website selling computer keyboards. If there were only a posted-price channel with price $p = \$100$, the seller would only sell to B_5, B_6, and B_7, for a revenue of $\$100 + \$100 + \$100 = \$300$. If he offers the two-unit auction and none of the high-valuation bidders participate, he will earn an additional \$20 (for a total of \$320), because B_4 and B_3 will win at the price of B_2's bid of \$10. If B_5 and B_6 choose to bid, they will win at a price of \$90, and the seller's total revenue will be $\$100 + \$90 + \$90 = \$280$, a decrease of \$20 with no additional sales. If, however, only B_6, among the high-valuation bidders, participates in the auction, then B_6 and B_4 will win the auction at a closing price of \$80, leading to \$200 of posted-price revenue (from B_5 and B_7) and \$160 of auction revenue (from B_6 and B_4), for a total of \$360.

This example illustrates that when high-valuation consumers do participate in the auction, there is the side benefit of an increase in the auction price. Hence, although the design of the auction should aim to discourage high-valuation consumers from bidding—which is equivalent to narrowing the time period in which high-valuation consumers choose to bid (reducing $\bar{t}$), as depicted in Figure 2—this objective is tempered by the positive effect high-valuation bidders have on auction prices. The seller also has two other related goals: to increase the number of units sold in the auction per unit time, and to increase the prices paid in the auction. To identify auction design strategies, we consider how the choice of q and T

Table 2 Potential Auction Design Strategies for High Arrival Rates

Goal:	Reduce $\bar{t}$	Increase auction price	Increase auction sales per unit of time
Strategy 1:	Large T	Large T	Large q
Strategy 2:	Small q	Small q	Small T

affects these three goals. We summarize this reasoning in Table 2.

Increasing T will increase the auction price because it increases the number of bidders who cannot buy at the posted price; i.e., it increases competition for the auctioned items (defined as the number of bidders per unit auctioned). A higher auction price means that $\bar{t}$ will be smaller because high-valuation consumers need a smaller delay cost to make the auction worthwhile. On the other hand, long auctions decrease the total auction sales per unit time because it takes longer to sell every unit auctioned. Short auctions have the opposite effect on each goal. Decreasing q is another way to increase competition in the auctions.

Small lot sizes will increase auction prices and as a result drive away high-valuation consumers, while at the same time, few items are sold via auction, and the auction sales are smaller. Large lot sizes can compensate for the negative aspects of long auctions, and short auctions can compensate for the negative aspects of small lot sizes. The two strategies described in Table 2 follow naturally. The seller should either (1) set one-unit auctions with length decreasing in the consumer-arrival rate, or (2) set long auctions with lot size increasing with the consumer-arrival rate. When the arrival rate is low, the seller's main concern is the auction price and cannibalization of sales at the posted price. In this case, the seller may need to set maximum length one-unit auctions to sell auctioned units above marginal cost, or it may even become suboptimal to add the auctions.

4.1. Numerical Experiments

In the following numerical experiments, we assume that consumer valuations follow a uniform distribution over $[\$0, \$100]$. Hence, we can use a closed-form expression for the expected value of the order statistics, $O\{x, y\}$, in our model. We use Equation (7) to calculate the seller's expected revenue for given λ, w, q, T, p, and $\bar{t}$. We vary the arrival rate λ between 0.5 and 60 consumers per day and vary the delay cost, w, between \$0.1 and \$5 per day. For each combination of λ and w, we determine the optimal values of q, T, and p.

Based on Proposition 1, the equilibrium auction participation strategy for high-valuation consumers consists of a threshold value, $\bar{t}$, given by the solution of the fixed-point equation $D(\bar{t}) = w\bar{t}$, where $D(\bar{t})$ is the expected auction discount over the posted price for a high-valuation bidder, assuming the number of additional high-valuation bidders and the number of low-valuation bidders follow the relevant Poisson distributions. However, to assume that consumers can actually solve $D(\bar{t}) = w\bar{t}$ for $\bar{t}$ means that we assume consumers are familiar with Poisson distributions, the properties of Poisson processes, and the manipulation of infinite sums (see Appendix 3 for the explicit expression of $D(\bar{t})$ using Poisson distributions for the number of bidders from each group). Although the seller is likely to have software tools that can aid in collecting data on consumer valuations and arrival patterns to determine the optimal values of the design parameters using Poisson distributions (i.e., using Equation (7)), it is reasonable to assume that the average consumer does not have such capabilities. In practice, it is most reasonable to expect that consumers will use some heuristic to determine their expected discount from the auction and therefore their participation threshold.

Because we do not know what heuristic individual bidders may be using, we propose a single heuristic to represent their behavior. We assume that consumers use expected (average) values rather than distributions of the number of bidders. In other words, when all other high-valuation consumers use a threshold $\bar{t}_h$, a high-valuation consumer estimates his expected auction discount assuming the number of low-valuation bidders is $\lambda \Pr(V < p)T$, and the number of additional high-valuation bidders is $\lambda \Pr(V \geq p)\bar{t}_h$. Because auction prices must be calculated using integer numbers of bidders and $\lambda \Pr(V < p)T$ and $\lambda \Pr(V \geq p)\bar{t}_h$ may be noninteger, we further assume that the high-valuation consumer interpolates between the nearest integer values for the expected number of low- and high-valuation bidders to determine the expected auction discount. Specifically, we have the following.

HEURISTIC 1. A high-valuation consumer evaluates his expected auction discount over the posted price as if the number of (other) high-valuation bidders (bidders with $V \geq p$) arriving in a period of length t is $\lceil \lambda \Pr(V \geq p)t \rceil$ with probability ρ and $\lfloor \lambda \Pr(V \geq p)t \rfloor$[3] with probability $(1-\rho)$, where $\rho = \lambda \Pr(V \geq p)t - \lfloor \lambda \Pr(V \geq p)t \rfloor$, and the number of low-valuation bidders arriving in a period of length t is $\lceil \lambda \Pr(V < p)t \rceil$ with probability γ and $\lfloor \lambda \Pr(V < p)t \rfloor$ with probability $(1-\gamma)$, where $\gamma = \lambda \Pr(V < p)t - \lfloor \lambda \Pr(V < p)t \rfloor$.

We do not claim that any bidders actually make the calculations described in the heuristic. However, we think that this heuristic captures the interactions among all the information potentially available to a bidder (except the nature of the stochastic process governing bidder arrivals), and therefore can represent what a bidder's heuristic might result in. We use $D_h(\bar{t})$ to denote the expected auction discount over the posted price for a high-valuation bidder using Heuristic 1 and assuming all other high-valuation consumers use the threshold $\bar{t}$. That is

$$\begin{aligned}
&D_h(\bar{t})\\
&= \rho(\bar{t})\gamma d(\lceil \lambda \Pr(V \geq p)\bar{t} + 1\rceil, \lceil \lambda \Pr(V < p)T \rceil) + (1-\rho(\bar{t}))\\
&\quad \cdot (1-\gamma) d(\lfloor \lambda \Pr(V \geq p)\bar{t} + 1\rfloor, \lfloor \lambda \Pr(V < p)T \rfloor)\\
&\quad + (1-\rho(\bar{t}))\gamma d(\lfloor \lambda \Pr(V \geq p)\bar{t} + 1\rfloor, \lceil \lambda \Pr(V < p)T \rceil)\\
&\quad + \rho(\bar{t})(1-\gamma)\\
&\quad \cdot d(\lceil \lambda \Pr(V \geq p)\bar{t} + 1\rceil, \lfloor \lambda \Pr(V < p)T \rfloor) \qquad (8)
\end{aligned}$$

where

$$d(x, y) = \begin{cases} p - O\{q - x + 1, y\} & \text{if } x \leq q \text{ and } x + y > q \\ 0 & \text{if } x > q \\ p - R & \text{if } x + y \leq q. \end{cases}$$

PROPOSITION 2. *When consumers use Heuristic* 1, *a unique* (*symmetric*) *equilibrium strategy for high-valuation consumers is given by the solution of the fixed-point equation,* $D_h(\bar{t}) = w\bar{t}$, *if* $D_h(T) < wT$, *and by* $\bar{t} = T$ *otherwise.*

The proof is in Appendix 1.

[3] $\lceil\,\rceil$ gives the closest larger integer, and $\lfloor\,\rfloor$ gives the closest smaller integer.

To summarize, given (w, λ), we use Equation (7) to calculate the seller's expected revenue, when consumers use Heuristic 1, for each triplet (q, T, p) to find the optimal dual-channel design. In Appendix 3, we present the results of numerical experiments analyzing performance when consumers do not use a heuristic but instead calculate the expected auction discount exactly, to determine the participation threshold $\bar{t}$. The results there are qualitatively similar to those derived using the above heuristic.

Recall that in our model, consumers with $V < p$ always participate in the auction because they have no other purchase options. Yet it is reasonable to assume that even such bidders will not join an auction if its duration is too long. In other words, we assume a maximum auction duration H, such that consumers ignore the auction until the time remaining in the auction is less than or equal to H. For the base case, we assume that $H = 168$ hours (seven days). Experiments with different values of H did not qualitatively change our results (see Appendix 2). To reduce the computational burden, we treat T as a discrete variable $T \in \{1 \text{ hour}, 2 \text{ hours}, \ldots, H \text{ hours}\}$. Table 3 summarizes our findings for $H = 7$ days.

Reinforcing our previous discussion, we see that the optimization results in one of the two cases we predicted, one-unit auctions with the length of the auction decreasing in the arrival rate or long auctions (seven days, the maximum length) with the size of the auction lot increasing in the arrival rate. The optimal setting depends on the delay cost per unit time incurred by high-valuation consumers and on the consumers' arrival rate.

When w is low, the only way to deter high-valuation consumers from bidding is to reduce the size of the auction lot. A long auction will not work, because if w is sufficiently low, high-valuation consumers will always choose to bid (regardless of the time they arrive during the auction). For small values of w, the seller should therefore offer one-unit auctions, and the length of the auctions should increase as the arrival rate decreases. When the delay cost, w, is high, it deters high-valuation consumers from bidding. The seller should set the auction length to the maximum and increase the size of the auction lot as the consumers' arrival rate increases. When arrival rates are low, the main concerns are the auction price and can-

Table 3 **The Optimal** $(q, T\,[\text{hr}], p\,[\$])$ **for Various Values of Consumer Arrival Rate,** λ**, and Delay Cost,** w**, when** $H = 168$ **Hours and** V **is Uniformly Distributed on** $[\$0, \$100]$

	w						
λ	\$0.1/day	\$0.5/day	\$1/day	\$2/day	\$3/day	\$4/day	\$5/day
0.5/day	No auction, $p = \$50$					1; *H*; \$56	1; *H*; \$56
1/day	**1; *H*; \$52**	1; *H*; \$53	1; *H*; \$53	1; *H*; \$53	1; *H*; \$54	1; *H*; \$54	1; *H*; \$55
2/day	**1; 134; \$52**	**1; 135; \$52**	2; *H*; \$53	2; *H*; \$53	2; *H*; \$54	3; *H*; \$56	3; *H*; \$57
5/day	**1; 53; \$52**	**1; 55; \$52**	6; *H*; \$52	7; *H*; \$54	8; *H*; \$55	8; *H*; \$57	9; *H*; \$58
10/day	**1; 27; \$52**	**1; 27; \$52**	13; *H*; \$52	16; *H*; \$54	18; *H*; \$56	19; *H*; \$58	20; *H*; \$59
20/day	**1; 13; \$52**	**1; 14; \$52**	30; *H*; \$52	35; *H*; \$54	38; *H*; \$56	39; *H*; \$58	40; *H*; \$59
30/day	**1; 9; \$52**	**1; 9; \$52**	47; *H*; \$52	54; *H*; \$54	58; *H*; \$57	60; *H*; \$58	62; *H*; \$60
40/day	**1; 7; \$52**	**1; 7; \$52**	**1; 7; \$52**	75; *H*; \$55	78; *H*; \$57	80; *H*; \$58	83; *H*; \$60
50/day	**1; 5; \$52**	**1; 5; \$52**	**1; 5; \$52**	94; *H*; \$55	98; *H*; \$57	100; *H*; \$58	104; *H*; \$60
60/day	**1; 4; \$52**	**1; 4; \$52**	**1; 4; \$52**	113; *H*; \$55	118; *H*; \$57	121; *H*; \$58	125; *H*; \$60

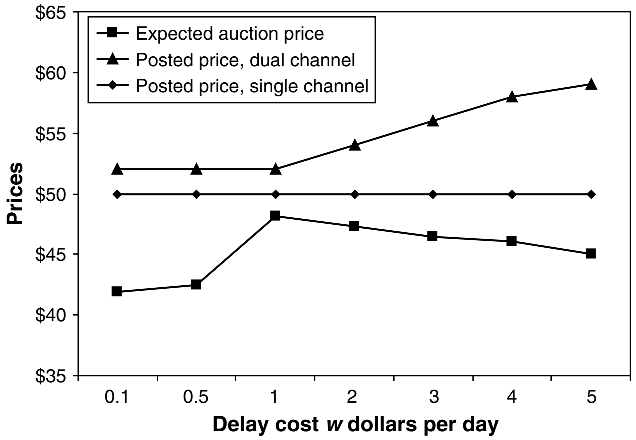
nibalization of the posted-price channel, so it becomes optimal to offer one-unit auction with the maximum length of seven days (which is an extreme case of each of the above two strategies). As the arrival rate decreases, the single channel (only posted price) might outperform the dual channel. As w increases, the dual channel outperforms the single channel even for smaller arrival rates. We can also see in Table 3 that the posted price is increasing as w increases and as λ increases, when auctions are long (H hours). When there is no auction, increasing the posted price from the optimal point leads to a loss of sales from those consumers who are priced out, which cancels any revenue gains from higher prices. When there is an auction, the consumer who was priced out by a price increase will not be completely lost because he may purchase in the auction, and his participation increases the auction price. As the website traffic, λ, increases, it becomes more attractive to have more sales via auctions. Increasing the posted price means that a larger proportion of the consumers are targeted by the auction channel. In Table 4, we see that as λ increases, the fraction of revenues coming from auctions increases.

Figure 3 shows how the posted price and the expected auction price change with the delay cost, w, when the arrival rate is 20 per day. In the range of w values where it is optimal to have one-unit auctions, the posted price is higher than the optimal price in the absence of auctions, and the expected auction price is nondecreasing in w. In the range of w values where it is optimal to have long auctions with multiple units of the item, the posted price is increasing in w, and the expected auction price is decreasing in w. This shows that as w increases, the two channels become more "separated;" that is, the gap between the posted price and the expected auction price increases. Hence, as w increases, both

Table 4 **Expected Auction Price and Fraction of Total Revenue Coming from the Auction Channel (Auction Price; Auction Revenue Fraction)**

	w						
λ	\$0.1/day	\$0.5/day	\$1/day	\$2/day	\$3/day	\$4/day	\$5/day
1/day	**\$36.97; 21%**	\$37.24; 21%	\$36.50; 20%	\$35.26; 19%	\$35.17; 19%	\$34.37; 19%	\$34.59; 19%
2/day	**\$42.11; 15%**	**\$41.85; 14%**	\$43.70; 24%	\$38.18; 23%	\$34.43; 22%	\$47.67; 32%	\$43.70; 31%
5/day	**\$42.05; 15%**	**\$42.26; 14%**	\$45.98; 30%	\$45.81; 35%	\$44.38; 38%	\$44.75; 38%	\$43.10; 40%
10/day	**\$42.25; 15%**	**\$42.17; 15%**	\$47.53; 34%	\$46.89; 41%	\$46.05; 44%	\$45.54; 45%	\$44.24; 45%
20/day	**\$41.87; 15%**	**\$42.52; 14%**	\$48.17; 40%	\$47.32; 45%	\$46.43; 47%	\$46.06; 47%	\$45.04; 46%
30/day	**\$42.21; 15%**	**\$42.21; 15%**	\$48.43; 42%	\$47.46; 46%	\$47.41; 48%	\$46.04; 48%	\$45.69; 48%
40/day	**\$42.52; 14%**	**\$42.52; 14%**	**\$42.52; 14%**	\$48.25; 49%	\$47.46; 49%	\$46.17; 48%	\$45.77; 48%
50/day	**\$41.42; 15%**	**\$41.42; 15%**	**\$41.42; 15%**	\$48.31; 49%	\$47.50; 49%	\$46.25; 48%	\$45.81; 48%
60/day	**\$40.99; 16%**	**\$40.99; 16%**	**\$40.99; 16%**	\$48.36; 49%	\$47.52; 49%	\$46.21; 48%	\$45.84; 48%

Figure 3 The Optimal Posted Price and the Expected Auction Price as a Function of the Delay Cost, w, for $\lambda = 20$/day



the delay cost and the expected auction discount increase.

When w gets large enough, the seller can offer more units in the auctions without a complete collapse of the posted-price channel. The result is that the increase in sales in the auction (due to the larger auction lot size) more than makes up for the reduction in posted-price sales (due to the higher posted price).

In Table 5, we show how the units sold are allocated among the different groups of consumers. When it is optimal to have short one-unit auctions, the number of units bought by high-valuation consumers in the auction is smaller than the number they buy at the posted price. When it is optimal to have long auctions with larger lot sizes, the auctions become responsible for a significant fraction of the sales to high-valuation consumers. We also see that more auction sales tend to go to the high-valuation consumers than to the low-valuation consumers and that the fraction of high-valuation consumers who buy the item via the auctions (given by the expected number of units bought by high-valuation consumers via auctions divided by the total expected number of units bought by high-valuation consumers) is increasing in λ. This means that more high-valuation consumers participate in auctions as λ increases. As the delay cost w increases (moving horizontally across the table), a higher proportion of auction sales go to the low-valuation consumers, because participating in the auction becomes too costly—in terms of delay—for the high-valuation consumers.

Figures 4 and 5 further show the changes in the optimal design, based on the numerical results in Table 3. From Figure 4, we see that when w is such that it is optimal to set the auction length to the maximum ($T = H$), the optimal lot size increases with the arrival rate at a relatively constant rate, enabling the seller to capture more consumers (to sell more units via auction per unit time). When the arrival rate is low, the size of the auction lot decreases to one unit. Figure 5 reveals that for small values of w, when it is optimal to have one-unit auctions, the optimal auction length is decreasing with the arrival rate. As the arrival rate increases, the shorter auctions enable the seller to capture more consumers per unit time.

4.2. Revenue Comparison

We now compare the revenue generated by four different selling approaches. The first approach is the

Table 5 Allocation of Items to Consumers per 24-Hour Period (Expected Number of Units Won by High Valuation; Expected Number of Units Won by Low Valuation; Expected Number Bought at Posted Price), for Various Values of the Consumer Arrival Rate, λ, and Delay Cost, w, when $H = 168$ Hours and V is Uniformly Distributed on [\$0, \$100]

	w						
λ	\$0.1/day	\$0.5/day	\$1/day	\$2/day	\$3/day	\$4/day	\$5/day
1/day	**0.09; 0.05; 0.39**	0.09; 0.06; 0.38	0.08; 0.06; 0.39	0.07; 0.07; 0.4	0.07; 0.07; 0.4	0.06; 0.08; 0.4	0.06; 0.08; 0.4
2/day	**0.11; 0.07; 0.85**	**0.11; 0.07; 0.85**	0.19; 0.1; 0.75	0.17; 0.11; 0.8	0.16; 0.13; 0.8	0.23; 0.2; 0.6	0.21; 0.22; 0.6
5/day	**0.29; 0.17; 2.1**	**0.27; 0.16; 2.1**	0.66; 0.2; 1.7	0.71; 0.29; 1.6	0.74; 0.4; 1.5	0.67; 0.48; 1.5	0.68; 0.61; 1.4
10/day	**0.56; 0.33; 4.2**	**0.56; 0.33; 4.2**	1.52; 0.34; 3.3	1.7 ; 0.58; 2.9	1.71; 0.86; 2.7	1.61; 1.1; 2.6	1.52; 1.33; 2.6
20/day	**1.16; 0.68; 8.4**	**1.08; 0.64; 8.5**	3.63; 0.65; 6.0	3.8 ; 1.2; 5.4	3.66; 1.77; 5.1	3.33; 2.25; 5.1	3.07; 2.65; 5.1
30/day	**1.68; 0.99; 12.7**	**1.68; 0.99; 12.7**	5.76; 0.95; 8.6	5.89; 1.82; 7.9	5.55; 2.73; 7.3	5.13; 3.45; 7.5	4.71; 4.15; 7.3
40/day	**2.15; 1.28; 17.06**	**2.15; 1.28; 17.0**	**2.15; 1.28; 17.0**	8.15; 2.56; 9.8	7.47; 3.67; 9.7	6.84; 4.59; 10.0	6.31; 5.55; 9.7
50/day	**3.0; 1.79; 21.0**	**3.0; 1.79; 21.0**	**3.0; 1.79; 21.0**	10.23; 3.2; 12.3	9.39; 4.61; 12.1	8.55; 5.73; 12.4	7.91; 6.95; 12.1
60/day	**3.75; 2.24; 25.0**	**3.75; 2.24; 25.0**	**3.75; 2.24; 25.0**	12.3; 3.84; 14.7	11.31; 5.55; 14.5	10.35; 6.93; 14.8	9.51; 8.35; 14.5

Figure 4 The Optimal Auction Lot Size as a Function of the Arrival Rate and Delay Cost, w

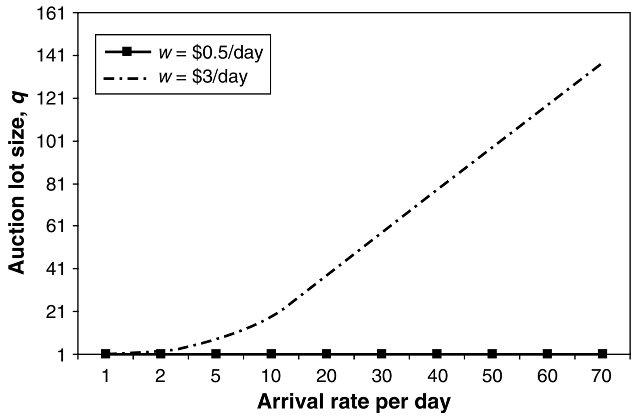
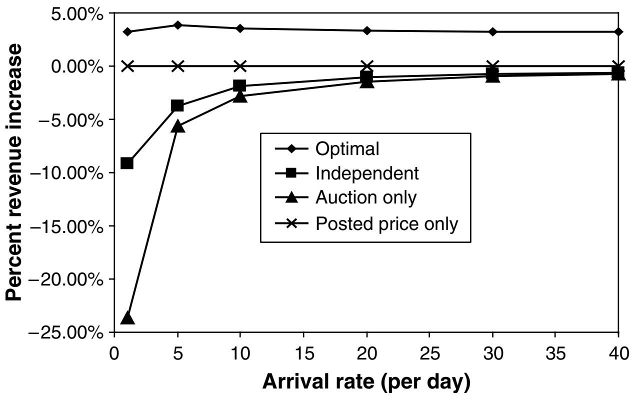


Figure 6 Revenue Increase Relative to Revenue from Using Only Posted Price, as a Function of the Arrival Rate, with Delay Cost $w = \$1$/day



optimal management of the dual channel as modeled in this paper. In the second approach, the auction and posted-price channels are managed independently. In this naive approach, the posted price is set as if there were no auction channel, and the auction channel design parameters T and q are selected as if there were no posted-price channel (see Appendix 4 for detailed results). The third approach is to use only an auction to sell goods, with T and q determined as in the independent approach. The fourth approach is to sell using only a posted-price channel.

In Figure 6, we plot the percentage revenue increase relative to the revenue from using only posted price for these approaches, as functions of the arrival rate λ, for $w = \$1$/day. In Figure 7, we plot the same as functions of the delay cost for $\lambda = 10$/day.

The dual channel's revenue is consistently greater than the posted-price revenue, which indicates that when managed optimally the dual channel can effectively serve the low-valuation consumers without cannibalizing the high-valuation consumers too much. On the other hand, when the channels are managed independently, there is too much cannibalization, and revenues are even worse than when there is just a posted-price channel.

In Figure 6, we see that when arrival rates are low, it is most important for the seller to optimize the two selling channels jointly. In Figure 7, we see that the seller's incentive to add auctions is greater when the delay cost perceived by consumers, w, is high.

Figure 5 The Optimal Auction Length as a Function of the Arrival Rate and Delay Cost, w

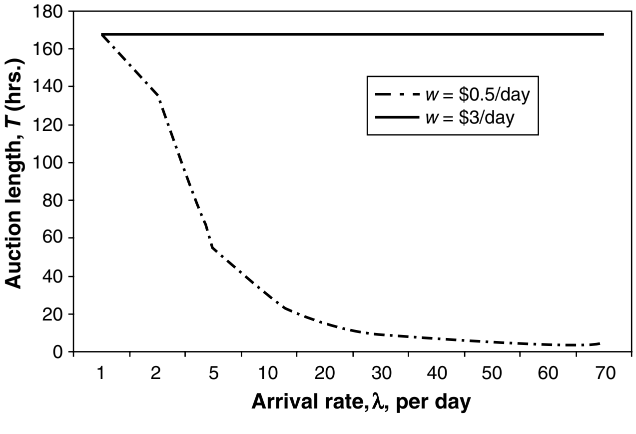
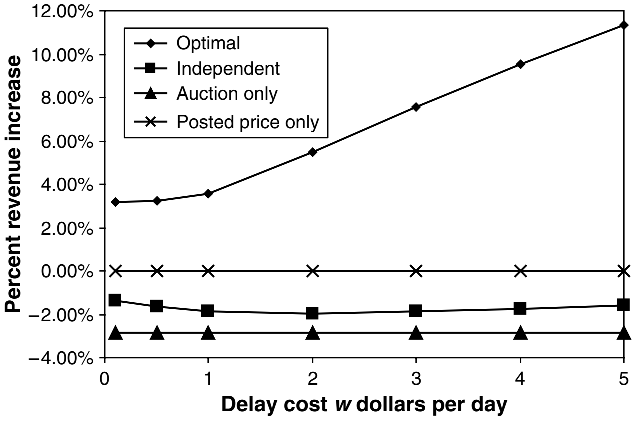


Figure 7 Revenue Increase Relative to Revenue from Only Posted Price as a Function of the Delay Cost, w, with Arrival Rate $\lambda = 10$/day



In such circumstances, more effective segmentation of consumers can be achieved because the seller can sell more via auction without losing too much posted-price revenue, and it becomes optimal to increase the auction length and lot size. Many firms have experimented with selling their products online, and many have experimented with online auctions. Our results suggest that unless they jointly manage their online offerings, these firms may find that auctions reduce their revenues. In Figure 7, we see potential revenue loss of 5% to 13% when comparing the independent-channel design with the optimal dual-channel design.

5. Concluding Remarks

It is possible to observe many firms selling the same or very similar goods online using posted prices and auctions simultaneously. Our paper explains this phenomenon by showing how posted price, auction lot size, and auction duration can be used to segment consumers between an online auction channel and a posted-price channel to increase a seller's revenue. This allows the auction to be used to capture customers who were priced out by the posted price while mitigating the effects of cannibalization of the posted-price channel. Numerical experiments show that the dual channel can significantly outperform a lone posted-price channel; we see revenue increases ranging from 2.4% to greater than 10%, which are considered significant improvements in retail sales. We show even greater benefits (4% to greater than 13%) over a naive approach to managing the two channels that optimizes each channel independently.

Balancing the need to avoid cannibalizing the posted-price channel with the opportunity to exploit the potential of the auction channel may seem daunting to a manager. Interestingly, our analysis shows that managing the dual channel optimally may be simpler than it seems. We find that as long as arrival rates are reasonably high, only two dominant strategies exist for managing the dual channel, and in both, the posted price is set higher than when there is no parallel auction. One strategy is to offer successive one-unit auctions parallel to the posted price. This strategy appears to be optimal for a wide range of parameter values and is indeed commonly observed in practice. The second strategy is to offer long auctions, and it becomes optimal as consumers' delay cost increases. The choice of the optimal strategy depends on the customer traffic to the website, λ, and the delay cost associated with the product being sold, w.

Our model contains several unique features. We model consumers as making their auction-participation decision using an estimate of their expected discount. Their decision is directly influenced by the quantity being auctioned and expectations about the number of other bidders. That is, bidders know that if they face greater competition in the auction, the discount over the posted-price channel will be smaller. The result is a very realistic and rich portrayal of bidder behavior.

There are a number of interesting areas for future research. By parameterizing our model on R, the seller's reserve price, we could identify the optimal reserve price as well as the posted price, lot size, and auction length. We suspect that the lot-size and auction-length variables already capture most of the effects of the reserve price. Such a result would be interesting in its own right. The heuristic we have proposed in Proposition 2 is optimistic in its assessment of the bidder's ability to estimate the auction discount. Future research could explore alternative heuristics and their impact on sellers' decisions.

Our model could also be extended to situations in which there is a finite supply that must be allocated between the two channels. Another extension would introduce competition between selling channels to address situations in which there are multiple auctioneers and posted-price sellers in the market. In all of these possible extensions, the underlying interaction between the auction lot size, auction length, and posted price introduced in this paper will play an important role. As demonstrated in Figures 6 and 7, failure to manage their interactions correctly can significantly reduce revenues.

Appendix 1

Proof of Lemma 1. The auction mechanism awards the items to the bidders with the q highest bids, and each pays a price equal to the highest losing bid. For each player i, define his type as his valuation V_i, and his action as his bid b_i. A strategy is a function from the type space $[\underline{v}, \bar{v}]$ to the action space, $B = [0, \infty)$.[4] See Gibbons (1992, ch. 3) for a detailed definition of static Bayesian games.

[4] When the current lowest bid required to win is listed, the action space can be bounded from below by that price (choosing not to

Define $b_{-i} = (b_1, b_2, \ldots, b_{i-1}, b_{i+1}, \ldots, b_N)$ as the vector of actions by the $N-1$ players besides player i. The payoff received by player i for each combination of actions that could be chosen by the other players, b_{-i}, is given by the following:[5]

$$U_i(b_i, b_{-i}; V_i) = \begin{cases} \theta[V_i - p_a(b_i, b_{-i})] + (1-\theta)[V_i - p] & \text{if } V_i \geq p \\ \theta[V_i - p_a(b_i, b_{-i})] & \text{if } V_i < p \end{cases} \quad (9)$$

where $p_a(b_i, b_{-i})$ is the resulting auction price, given by the $q+1$ highest bid when there are more than q bids, and by the seller's reserve price otherwise, and θ is the probability player i gets an item in the auction. By definition, $\theta \in [0, 1]$, and clearly $\theta = 1$ if $b_i > p_a(b_i, b_{-i})$, and $\theta = 0$ if $b_i < p_a(b_i, b_{-i})$ or if $b_i = p_a(b_i, b_{-i})$ and b_i is the only bid that equals the auction closing price. When $b_i = p_a(b_i, b_{-i})$ and b_i is not the only bid that equals the closing price, the specific expression for θ depends on the tie-breaking rule in use, but regardless of the tie-breaking rule, it is still true that $\theta \in [0, 1]$, and this is the only property of θ used in the proof.

Next, we show that when player i values the item for more than its posted price, $V_i \geq p$, he can do no better than bidding p. That is, for any other action $b'_i \neq p$, for any number of bidders N and combination of other bidders' actions b_{-i}, $U_i(p, b_{-i}; V_i) \geq U_i(b'_i, b_{-i}; V_i)$, with strict inequality for some b_{-i}.

We divide the space of feasible b_{-i} into three exclusive sets (columns in Table 6): the first set includes instances of b_{-i} that together with $b_i = p$ result in an auction price that is higher than p; the second set includes instances of b_{-i} that together with $b_i = p$ result in an auction price that equals p; and the last set includes instances of b_{-i} that together with $b_i = p$ result in an auction price that is lower than p. Table 6 shows the payoff matrix for a player of type $V_i \geq p$.

For any $\Delta > 0$ and any b_{-i}, $U_i(p, b_{-i}; V_i) \geq U_i(p+\Delta, b_{-i}; V_i)$ and $U_i(p, b_{-i}; V_i) \geq U_i(p-\Delta, b_{-i}; V_i)$ and there is some b_{-i} for which these inequalities are strict. A detailed analysis follows.

Payoffs when $b_i = p$

- When b_{-i} is such that $p_a(p, b_{-i}) > p$, player i does not win the item in the auction and purchases it for the posted price, with payoff $V_i - p$.
- When b_{-i} is such that $p_a(p, b_{-i}) = p$, player i wins the item in the auction with probability $0 \leq \theta \leq 1$. If he wins the item, he pays p, and if he does not win, he purchases it for p. Either way his payoff is $V_i - p$.
- When b_{-i} is such that $p_a(p, b_{-i}) < p$, player i wins the item in the auction with payoff $V_i - p_a(p, b_{-i}) > V_i - p$.

Payoffs when $b'_i = p + \Delta$

- When b_{-i} is such that $p_a(p, b_{-i}) > p$, player i's payoff as a function of b_{-i} is given by $\theta[V_i - p_a(p+\Delta, b_{-i})] + [1-\theta][V_i - p]$, where $p_a(p+\Delta, b_{-i}) \geq p_a(p, b_{-i}) > p$ and $\theta \in [0, 1]$. The payoff thus is never higher than the payoff when he bids p. When b_{-i} is such that $p_a(p, b_{-i}) < p + \Delta$, player i wins the item in the auction and pays $p_a(p+\Delta, b_{-i}) \geq p_a(p, b_{-i}) > p$, so his payoff is less than his payoff from bidding p.
- When b_{-i} is such that $p_a(p, b_{-i}) = p$, player i's payoff as a function of b_{-i} is given by $\theta[V_i - p_a(p+\Delta, b_{-i})] + [1-\theta][V_i - p]$, where $p_a(p+\Delta, b_{-i}) \geq p_a(p, b_{-i}) = p$ and $\theta \in [0, 1]$. The payoff is thus never higher than the payoff when he bids p. Furthermore, for b_{-i} such that $p + \Delta > p_a(p+\Delta, b_{-i}) > p$, player i wins the item in the auction and pays $p_a(p+\Delta, b_{-i}) > p$, so his payoff is less than his payoff from bidding p.
- When b_{-i} is such that $p_a(p, b_{-i}) < p$, player i wins the item in the auction with payoff $v_i - p_a(p, b_{-i})$, the same payoff as when he bids p, since $p_a(p+\Delta, b_{-i}) = p_a(p, b_{-i})$.

Notice that $U_i(p, b_{-i}; V_i) \geq U_i(p+\Delta, b_{-i}; V_i)$ for every feasible vector b_{-i}, with strict inequality for some instances of b_{-i}. We conclude that $b'_i = p + \Delta$ is weakly dominated by $b_i = p \; \forall \Delta > 0$.

Payoffs when $b'_i = p - \Delta$

- When b_{-i} is such that $p_a(p, b_{-i}) > p$, player i does not win the auction and thus purchases the item for the posted price with payoff of $V_i - p$, which is the same as when he bids p.
- When b_{-i} is such that $p_a(p, b_{-i}) = p$, player i does not win the auction and thus purchases the item for the posted price with payoff of $V_i - p$, which is the same as when he bids p.
- When b_{-i} is such that $p_a(p, b_{-i}) < p$, player i's payoff as a function of b_{-i} is given by $\theta[V_i - p_a(p-\Delta, b_{-i})] + [1-\theta] \cdot [V_i - p]$, where $\theta > 0$ only if $p_a(p-\Delta, b_{-i}) = p_a(p, b_{-i})$ (if player i lowers the auction price by bidding less than p, he will be the highest losing bid). The payoff thus is never higher than the payoff from bidding p and is strictly lower for b_{-i} such that $\theta < 1$.

We note that $U_i(p, b_{-i}; V_i) \geq U_i(p-\Delta, b_{-i}; V_i)$ with strict inequality for some instances of b_{-i}. We conclude that $b'_i = p - \Delta$ is weakly dominated by $b_i = p$. Because the action $b_i = p$ weakly dominates every other action in B, it is a weakly dominant action for a bidder with $V_i > p$ (Mas-Colell et al. 1995, p. 238).

In a similar way, we can prove that $b_i = V_i$ is a weakly dominant action when $V_i < p$. Table 7 shows that for any number of bidders, N, and for any combination of bidders' actions, b_{-i}, player i cannot do better than bidding V_i by bidding $V_i - \Delta$ or $V_i + \Delta$, for any $\Delta > 0$, and for some instances of b_{-i} he does strictly worse.

bid dominates a bid lower than the current lowest bid required to win).

[5] We do not include any cost of delay because once the consumer decides to bid, this cost is sunk. Here we assume that the decision to bid has already been made, and the search is for a bidding strategy.

Table 6 **Payoff Matrix for a Player of Type $V \geq p$**

	b_{-i}: $p_a(p, b_{-i}) > p$	b_{-i}: $p_a(p, b_{-i}) = p$	b_{-i}: $p_a(p, b_{-i}) < p$
$b_i = p - \Delta$	$V_i - p$	$V_i - p$	$\theta[V_i - p_a(p, b_{-i})] + [1-\theta][V_i - p]$ with $\theta < 1$ for some b_{-i}
$b_i = p$	$V_i - p$	$V_i - p$	$V_i - p_a(p, b_{-i})$
$b_i = p + \Delta$	$\theta[V_i - p_a(p+\Delta, b_{-i})] + [1-\theta][V_i - p]$ with $\theta > 0$ for some b_{-i}	$\theta[V_i - p_a(p+\Delta, b_{-i})] + [1-\theta][V_i - p]$ with $\theta > 0$ and $p_a(p+\Delta, b_{-i}) > p$ for some b_{-i}	$V_i - p_a(p, b_{-i})$

Since the strategy

$$b(V) = \begin{cases} V & \text{for } V < p \\ p & \text{for } V \geq p \end{cases}$$

satisfies $U_i(b(V_i), b_{-i}; V_i) \geq U_i(b'(V_i), b_{-i}; V_i)$ for every feasible vector b_{-i}, with strict inequality for some instances of b_{-i}, it is a weakly dominant strategy. □

Proof of Proposition 1. Decision rule (5), "participate in the auction if and only if

$$\Pr(win \mid p)p - E[auction\_payment \mid p] \geq wt^e\text{"}$$

is derived by rearranging the condition $U_+^A(V, t^e) \geq U_+^B(V, t^e)$ and assuming that bidders bid according to the strategy specified in Lemma 1. According to decision rule (5), a high-valuation consumer chooses to participate in the auction if and only if his expected discount from participating in the auction exceeds his delay cost.

We define $Q(x)$ as the probability consumer i wins the auction under some tie-breaking rule when there are x other high-valuation bidders. Consumer i's expected discount from participating in the auction (the LHS in the above decision rule) is given by

$$\begin{aligned} &p\Pr(win \mid p) - E[auction\_payment \mid p] \\ &= p\left(\sum_{x=0}^{q-1}\Pr(N_{/i}^+ = x) + \sum_{x=q}^{\infty} Q(x)\Pr(N_{/i}^+ = x)\right) \\ &\quad -\left(p\sum_{x=q}^{\infty} Q(x)\Pr(N_{/i}^+ = x) + R\sum_{x=0}^{q-1}\sum_{y=0}^{q-x-1}\Pr(N_{/i}^+ = x)\Pr(N^- = y)\right. \\ &\quad + \sum_{x=0}^{q-1}\sum_{y=q-x}^{\infty}\Pr(N_{/i}^+ = x)\Pr(N^- = y) \\ &\quad \left.\cdot O\{q-(x+1)+1, y\}\right) \end{aligned} \tag{10}$$

where $N_{/i}^+$ is the number of high-valuation bidders besides consumer i who have decided to participate in the auction, and N^- is the number of low-valuation bidders. Notice that the particular tie-breaking rule in use does not affect the results because all the terms that depend on the tie-breaking rule cancel out. Canceling terms, we see that a high-valuation consumer finds it optimal to participate in the auction iff

$$\begin{aligned} &p\left(\sum_{x=0}^{q-1}\Pr(N_{/i}^+ = x)\right) - R\sum_{x=0}^{q-1}\sum_{y=0}^{q-x-1}\Pr(N_{/i}^+ = x)\Pr(N^- = y) \\ &\quad - \sum_{x=0}^{q-1}\sum_{y=q-x}^{\infty}\Pr(N_{/i}^+ = x)\Pr(N^- = y)O\{q-x, y\} \geq wt^e. \end{aligned} \tag{11}$$

Next, we prove that there is a unique symmetric equilibrium in which all high-valuation consumers use a threshold policy to choose between buying the item for the posted price and participating in the auction. That is, we show that there is a unique value $\bar{t}$ so that if all other high-valuation consumers use the threshold $\bar{t}$ to choose between buying the item for the posted price and bidding in the auction, then any other high-valuation customer will find that his best response is to use the same threshold as well.

Notice that the LHS in (11) does not depend on the valuation of the consumer making the decision, nor on the

Table 7 **Payoff Matrix for a Player of Type $V < p$**

	b_{-i}: $p_a(V_i, b_{-i}) > V_i$	b_{-i}: $p_a(V_i, b_{-i}) = V_i$	b_{-i}: $p_a(V_i, b_{-i}) < V_i$
$b_i = V_i - \Delta$	0	0	$\theta[V_i - p_a(V_i, b_{-i})]$ with $\theta < 1$ for some b_{-i}
$b_i = V_i$	0	0	$V_i - p_a(V_i, b_{-i})$
$b_i = V_i + \Delta$	$\theta[V_i - p_a(v_i + \Delta, b_{-i})] \leq 0$ with $\theta > 0$ for some b_{-i}	$\theta[V_i - p_a(V_i + \Delta, b_{-i})] \leq 0$	$V_i - p_a(V_i, b_{-i})$

remaining time of the auction he observes upon arrival, t^e, because he bids $b = p$ regardless of his V and t^e (Lemma 1).

If all other high-valuation consumers use the threshold $\bar{t}$, then the number of other high-valuation bidders, $N^+_{/i}$, has a Poisson distribution with rate $\lambda\Pr(V \geq p)\bar{t}$,[6] and the number of low-valuation bidders, N^-, has a Poisson distribution with rate $\lambda\Pr(V < p)T$.

The LHS of (11) is thus a function of $\bar{t}$,

$$D(\bar{t}) = \sum_{x=0}^{q-1} \frac{e^{-\lambda\Pr(V\geq p)\bar{t}}(\lambda\Pr(V \geq p)\bar{t})^x}{x!}\bigg(p - R\sum_{y=0}^{q-x-1}\Pr(N^- = y) - \sum_{y=q-x}^{\infty}\Pr(N^- = y)O\{q-x, y\}\bigg), \tag{12}$$

and the consumer under consideration chooses to participate in the auction iff

$$D(\bar{t}) \geq wt^e, \tag{13}$$

where

$$\Pr(N^- = y) = \frac{e^{-\lambda\Pr(V<p)T}(\lambda\Pr(V < p)T)^y}{y!}$$

and is not a function of $\bar{t}$.

We can see that the expected discount from participating in the auction is continuous in the threshold used by all other high-valuation bidders; i.e., $D(\bar{t})$ is continuous in $\bar{t}$, since it consists of linear combination of polynomial and exponential functions of $\bar{t}$. $O\{q-x, y\}$, the expected value of the $q-(x+1)+1$ order statistic of y draws from the consumer valuations distribution truncated on $[\underline{v}, p]$, is not a function of $\bar{t}$ and can be treated as a constant. We next show that $D(\bar{t})$ is decreasing in $\bar{t}$.

Denote

$$f(x, \bar{t}) = \frac{e^{-\lambda\Pr(V\geq p)\bar{t}}(\lambda\Pr(V \geq p)\bar{t})^x}{x!} \quad \text{and}$$

$$g(x) = p - R\sum_{y=0}^{q-x-1}\Pr(N^- = y) - \sum_{y=q-x}^{\infty}\Pr(N^- = y)O\{q-x, y\}.$$

Then

$$D(\bar{t}) = \sum_{x=0}^{q-1} f(x, \bar{t})g(x).$$

In the following, we suppress the dependence of f on $\bar{t}$.

First, we show that $g(x)$, the expected discount from the auction when there are x other high-valuation bidders, is decreasing in x; that is, we show that $g(x+1) \leq g(x)$.

$$\begin{aligned} g(x+1) - g(x) &= R\Pr(N^- = q-x-1) - \Pr(N^- = q-x-1)O\{q-x-1, y\} \\ &= \Pr(N^- = q-x-1)(R - O\{q-x-1, y\}) \leq 0, \end{aligned}$$

where the last inequality holds because $O\{q-x-1, y\} \geq R$, since no bids are less than R. In addition, $g(x) > 0$ for every value of x because $R < p$ and $O\{q-x, y\} < p$.

Next, using the fact that

$$\frac{\partial f(x, \bar{t})}{\partial \bar{t}} = \begin{cases} \lambda\Pr(V \geq p)(f(x-1, \bar{t}) - f(x, \bar{t})) & \text{if } x > 0 \\ -\lambda\Pr(V \geq p)e^{-\lambda\Pr(V\geq p)\bar{t}} & \text{if } x = 0 \end{cases}$$

we take the derivative of $D(\bar{t})$ with respect to $\bar{t}$:

$$\begin{aligned} \frac{\partial D(\bar{t})}{\partial \bar{t}} &= \frac{\partial\sum_{x=0}^{q-1} f(x)g(x)}{\partial \bar{t}} \\ &= \frac{\partial f(0)g(0)}{\partial \bar{t}} + \lambda\Pr(V \geq p)\sum_{x=1}^{q-1} g(x)(f(x-1) - f(x)) \\ &= \lambda\Pr(V \geq p)\bigg(-g(0)e^{-\lambda\Pr(V\geq p)\bar{t}} + \sum_{x=1}^{q-2} f(x)(g(x+1) - g(x)) \\ &\qquad + g(1)f(0) - g(q-1)f(q-1)\bigg) < 0. \end{aligned}$$

The last inequality holds because $g(0) > 0$, $g(q-1)f(q-1) > 0$, $g(x+1) < g(x)$ as shown above, and

$$g(1)f(0) = g(1)e^{-\lambda\Pr(V\geq p)\bar{t}} < g(0)e^{-\lambda\Pr(V\geq p)\bar{t}}.$$

Notice that $t^e \leq T$, because a bidder cannot arrive to an auction that has not started, so we only consider values of $\bar{t}$ that do not exceed T. If $D(T) \geq wT$, then the unique symmetric threshold equilibrium is given by the threshold T. If, on the contrary, there is a symmetric threshold equilibrium s such that $s < T$, then a high-valuation consumer who arrives with T time units remaining in the auction has best response to participate in the auction. Such a consumer has less competition than if all high-valuation consumers use T as a threshold, and so his discount is greater than $D(T)$, which, by assumption, is greater than his delay cost wT. Hence, $s < T$ cannot be a symmetric threshold equilibrium when $D(T) \geq wT$. If all other high-valuation consumers use the threshold T, then because $D(T) \geq wT$, a high-valuation consumer's best response is to use the threshold T.

If $D(T) < wT$, then T cannot be a symmetric threshold equilibrium, since if all other high-valuation consumers use the threshold T, a high-valuation consumer's best response

[6] We assume that the consumer making the decision uses arrival rate λ, i.e., he does not use a lower arrival rate to account for his own arrival. This makes sense due to the "memory-less" nature of the exponential distribution (at each point of time the distribution of the time remaining until the next arrival stays the same) and due to PASTA—Poisson Arrivals See Time Average.

is to use a threshold smaller than T. However, we now show that if $D(T) < wT$, then there is a unique symmetric threshold equilibrium, $0 < \bar{t} < T$.

Assume $D(T) < wT$ and that all other high-valuation consumers use the threshold $0 < \bar{t} < T$. A high-valuation consumer chooses to participate in the auction iff $D(\bar{t}) \geq wt^e$, and since $D(\bar{t})$ is not a function of t^e, the consumer uses the threshold t^*, given by the solution of $D(\bar{t}) = wt^*$. $D(\bar{t}) > 0$, since there is a positive probability that the expected auction price is strictly less than p, and so t^* has to be positive. In addition, since $D(\bar{t})$ is continuous and strictly decreasing in $\bar{t}$, there exists a unique value of $\bar{t}$ such that $D(\bar{t}) = w\bar{t}$, and this value has to be smaller than T, because $D(T) < wT$. That value of $\bar{t}$ is the unique symmetric threshold equilibrium. □

Proof of Proposition 2. To simplify notation, we define $\lambda^+ = \lambda \Pr(V \geq p)t$ and $\lambda^- = \lambda \Pr(V < p)T$ (note that λ^+ is continuous and strictly increasing in t, and λ^- is independent of t). $D_h(t)$, the expected auction discount over the posted price for a high-valuation bidder who uses Heuristic 1 and assumes that all other high-valuation consumers use the threshold t, is given by

$$\begin{aligned}&\rho(t)\gamma d(\lceil \lambda \Pr(V \geq p)t+1\rceil, \lceil \lambda \Pr(V < p)T\rceil)\\&\quad+(1-\rho(t))(1-\gamma)d(\lfloor \lambda \Pr(V \geq p)t+1\rfloor, \lfloor \lambda \Pr(V < p)T\rfloor)\\&\quad+(1-\rho(t))\gamma d(\lfloor \lambda \Pr(V \geq p)t+1\rfloor, \lceil \lambda \Pr(V < p)T\rceil)\\&\quad+\rho(t)(1-\gamma)d(\lceil \lambda \Pr(V \geq p)t+1\rceil, \lfloor \lambda \Pr(V < p)T\rfloor) \qquad (14)\end{aligned}$$

where $d(x, y)$ is the expected discount over the posted price for a high-valuation consumer who participates in an auction with a total of x high-valuation bidders (including himself) and y low-valuation bidders, as given in Equation (15):

$$d(x,y)=\begin{cases}p-O\{q-x+1,y\} & \text{if } x\leq q \text{ and } x+y>q\\0 & \text{if } x>q\\p-R & \text{if } x+y\leq q.\end{cases} \qquad (15)$$

Because we use the uniform distribution for each consumer's valuation of the item, in our numerical experiments

$$O(q-x+1,y)=p\frac{y-(q-x)}{y+1}$$

(see Pinker et al. 2003).

We next show that there is a unique (symmetric) equilibrium in which all high-valuation consumers use the threshold $\bar{t}_h$. That is, when $D_h(T) < wT$, there is a unique value $t < T$ satisfying $D_h(t) = wt$, and we denote it by $\bar{t}_h$, and when $D_h(T) \geq wT$, $\bar{t}_h = T$.

As in the proof of Proposition 1, since wt is strictly increasing in t, it is sufficient to show that $D_h(t)$ is continuous and nonincreasing in t and that $D_h(t) > 0$. We examine $D_h(t)$ over five ranges of t values, starting from the right.

Range 1: $t \geq q/[\lambda \Pr(V > p)]$. In this range, we have $\lfloor \lambda^+ + 1\rfloor > q$, and so $D_h(t)$ equals 0.

Range 2: $(q-1)/[\lambda \Pr(V \geq p)] < t < q/[\lambda \Pr(V \geq p)]$. In this range, $\lceil \lambda^+ + 1\rceil = q+1 > q$, but $\lfloor \lambda^+ + 1\rfloor = q$, so $D_h(t)$ is given by

$$\begin{aligned}&\gamma(1-\rho(t))d(\lfloor \lambda^+ +1\rfloor, \lceil \lambda^-\rceil)+(1-\gamma)(1-\rho(t))d(\lfloor \lambda^+ +1\rfloor, \lfloor \lambda^-\rfloor)\\&\quad=\gamma(1-(\lambda^+ - \lfloor \lambda^+\rfloor))\frac{p}{\lceil \lambda^-\rceil+1}\\&\qquad+(1-\gamma)(1-(\lambda^+ - \lfloor \lambda^+\rfloor))\frac{p}{\lfloor \lambda^-\rfloor+1},\end{aligned}$$

which is continuous and linearly decreasing in t (since λ^+ is linearly increasing in t and $\lfloor \lambda^+\rfloor = q-1$ for this entire range of t values). We next show that when t approaches the upper bound of this range, the expected auction discount approaches the value of the expected auction discount at the lower bound of Range 1, and that when t approaches the lower bound of this range, the expected auction discount approaches its value at the upper bound of Range 3.

$$\begin{aligned}&\lim_{t\to q/[\lambda \Pr(V\geq p)] \text{ from below}} D_h(t)\\&\quad=\lim_{\lambda^+\to q}\left\{\gamma(1-(\lambda^+-(q-1)))\left(\frac{p}{\lceil \lambda^-\rceil+1}\right)\right.\\&\qquad\left.+(1-\gamma)(1-(\lambda^+-(q-1)))\left(\frac{p}{\lfloor \lambda^-\rfloor+1}\right)\right\}=0.\end{aligned}$$

Using $\lim_{\lambda^+\to\lceil \lambda^+\rceil}(\lambda^+ - \lfloor \lambda^+\rfloor) = 1$,

$$\begin{aligned}&\lim_{t\to (q-1)/[\lambda \Pr(V\geq p)] \text{ from above}} D_h(t)\\&\quad=\lim_{\lambda^+\to q-1}\left\{\gamma(1-(\lambda^+-(q-1)))\left(\frac{p}{\lceil \lambda^-\rceil+1}\right)\right.\\&\qquad\left.+(1-\gamma)(1-(\lambda^+-(q-1)))\left(\frac{p}{\lfloor \lambda^-\rfloor+1}\right)\right\}\\&\quad=\frac{\gamma p}{\lceil \lambda^-\rceil+1}+\frac{(1-\gamma)p}{\lfloor \lambda^-\rfloor+1}=D_h\left(\frac{q-1}{\lambda \Pr(V\geq p)}\right).\end{aligned}$$

Range 3: $\max(0, (q-\lfloor \lambda^-\rfloor)/[\lambda \Pr(V \geq p)]) \leq t \leq (q-1)/[\lambda \Pr(V \geq p)]$. In this range, $\lceil \lambda^+ + 1\rceil \leq q$ and $\lfloor \lambda^+ + 1\rfloor + \lfloor \lambda^-\rfloor > q$. Therefore, $D_h(t)$ is given by

$$\begin{aligned}&\gamma(\lambda^+ - \lfloor \lambda^+\rfloor)\left(\frac{p(q-\lceil \lambda^+\rceil)}{\lceil \lambda^-\rceil+1}\right)\\&\quad+\gamma(1-(\lambda^+ - \lfloor \lambda^+\rfloor))\left(\frac{p(q-\lfloor \lambda^+\rfloor)}{\lceil \lambda^-\rceil+1}\right)\\&\quad+(1-\gamma)(\lambda^+ - \lfloor \lambda^+\rfloor)\left(\frac{p(q-\lceil \lambda^+\rceil)}{\lfloor \lambda^-\rfloor+1}\right)\\&\quad+(1-\gamma)(1-(\lambda^+ - \lfloor \lambda^+\rfloor))\left(\frac{p(q-\lfloor \lambda^+\rfloor)}{\lfloor \lambda^-\rfloor+1}\right)\\&=\left(\frac{\gamma p}{\lceil \lambda^-\rceil+1}+\frac{(1-\gamma)p}{\lfloor \lambda^-\rfloor+1}\right)(q-\lambda^+).\end{aligned}$$

In this range of t values, $D_h(t)$ is continuous and linearly decreasing in t because λ^+ is linearly increasing and continuous in t.

Range 4: $\max(0, (q - \lceil\lambda^-\rceil - 1)/[\lambda \Pr(V \ge p)]) < t < \max(0, (q - \lfloor\lambda^-\rfloor)/[\lambda \Pr(V \ge p)])$. It is easy to show that in this range of t values, $D_h(t)$ is continuous and linearly decreasing in t whenever λ^+ is noninteger. Hence, here we focus on showing that $D_h(t)$ is continuous when λ^+ is an integer for t in the above range, and at the upper and lower bounds of this range. When λ^+ approaches an integer from below, $\lim(\lambda^+ - \lfloor\lambda^+\rfloor) = 1$, so $\lim \rho(t) = 1$. Therefore,

$$\begin{aligned}
&\lim_{t \xrightarrow{below} (q-\lfloor\lambda^-\rfloor)/[\lambda \Pr(V\ge p)]} D_h(t)\\
&\quad = \gamma\left(p\frac{\lfloor\lambda^-\rfloor}{\lceil\lambda^-\rceil+1}\right) + (1-\gamma)\left(p\frac{\lfloor\lambda^-\rfloor}{\lfloor\lambda^-\rfloor+1}\right)\\
&\quad = \left(\frac{\gamma p}{\lceil\lambda^-\rceil+1} + \frac{(1-\gamma)p}{\lfloor\lambda^-\rfloor+1}\right)\lfloor\lambda^-\rfloor\\
&\quad = D_h\left(\frac{q-\lfloor\lambda^-\rfloor}{\lambda \Pr(V\ge p)}\right).
\end{aligned}$$

When λ^+ approaches an integer from above, $\lim(\lambda^+ - \lfloor\lambda^+\rfloor) = 0$, so $\lim \rho(t) = 0$. Therefore,

$$\begin{aligned}
&\lim_{t \xrightarrow{above} (q-\lceil\lambda^-\rceil-1)/[\lambda \Pr(V\ge p)]} D_h(t)\\
&\quad = \lim_{t \xrightarrow{above} (q-\lceil\lambda^-\rceil-1)/[\lambda \Pr(V\ge p)]} \rho(t)\gamma d(\lceil\lambda^+ + 1\rceil, \lceil\lambda^-\rceil)\\
&\qquad + (1-\rho(t)\gamma)(p-R) = P - R.
\end{aligned}$$

If λ^- is an integer, then there are no t inside Range 4 such that λ^+ is integer. When λ^- is not an integer, there is only one value of t, in Range 4, for which λ^+ is an integer, namely, $(q - \lceil\lambda^-\rceil)/[\lambda \Pr(V \ge p)]$, and

$$D_h\left(\frac{q-\lceil\lambda^-\rceil}{\lambda \Pr(V\ge p)}\right) = \gamma d(q - \lceil\lambda^-\rceil + 1, \lceil\lambda^-\rceil) + (1-\gamma)(p-R).$$

We show now that $D_h(t)$ is continuous at this value of t:

$$\begin{aligned}
&\lim_{t \xrightarrow{below} (q-\lceil\lambda^-\rceil)/[\lambda \Pr(V\ge p)]} D_h(t)\\
&\quad = \lim_{\lambda^+ \xrightarrow{below} q-\lceil\lambda^-\rceil} \gamma d(\lceil\lambda^+ + 1\rceil, \lceil\lambda^-\rceil) + (1-\gamma) d(\lceil\lambda^+ + 1\rceil, \lfloor\lambda^-\rfloor)\\
&\quad = \gamma d(q - \lceil\lambda^-\rceil + 1, \lceil\lambda^-\rceil) + (1-\gamma)(p-R)\\
&\quad = D_h\left(\frac{q-\lceil\lambda^-\rceil}{\lambda \Pr(V\ge p)}\right),\\
&\lim_{t \xrightarrow{above} (q-\lceil\lambda^-\rceil)/[\lambda \Pr(V\ge p)]} D_h(t)\\
&\quad = \lim_{\lambda^+ \xrightarrow{above} q-\lceil\lambda^-\rceil} (1-\gamma) d(\lfloor\lambda^+ + 1\rfloor, \lfloor\lambda^-\rfloor) + \gamma d(\lfloor\lambda^+ + 1\rfloor, \lceil\lambda^-\rceil)\\
&\quad = \gamma d(q - \lceil\lambda^-\rceil^{+}1, \lceil\lambda^-\rceil) + (1-\gamma)(p-R)\\
&\quad = D_h\left(\frac{q-\lceil\lambda^-\rceil}{\lambda \Pr(V\ge p)}\right).
\end{aligned}$$

Range 5: $0 \le t \le \max(0, (q - \lceil\lambda^-\rceil - 1)/\lambda^-)$. In this range, $D_h(t) = (p - R)$, since for such values of t, the total number of bidders does not exceed q in each of the four terms in $D_h(t)$ and $d(x, y) = p - R$ when $(x + y) \le q$.

Because $D_h(t)$ is continuous and constant or linearly decreasing in t in each of the above five ranges of t values, and continuous at the edges of each range, we conclude that $D_h(t)$ is continuous and nonincreasing in t for every $t \ge 0$. In addition, $D_h(0) = (p - R) > 0$. Hence, $D_h(t) = wt$ is a fixed-point equation with a unique solution. The rest of the proof is similar to the proof of Proposition 1. □

Appendix 2. Additional Numerical Results

Tables 8 and 9 present the optimal design of the dual channel, when the participation threshold used by high-valuation consumers is derived from Proposition 2, for $H = 5$ days and $H = 3$ days (the base case of $H = 7$ days is in §4). The results are very similar to those in Table 3 except that the auctions are shorter, with smaller lot sizes, and the posted prices are lower.

Table 8 **The Optimal (q, T [hr], p [$]) for Various Parameter Values for $H = 5$ Days and $\bar{t}$ Derived from Proposition 2**

	w						
λ	$0.1/day	$0.5/day	$1/day	$2/day	$3/day	$4/day	$5/day
0.5/day	No auction, $p = \$50$						
1/day	**1; *H*; $52**	1; *H*; $53	1; *H*; $53	1; *H*; $54	1; *H*; $54	1; *H*; $54	1; *H*; $55
2/day	**1; *H*; $52**	**1; *H*; $52**	1; *H*; $52	1; *H*; $52	1; *H*; $53	1; *H*; $53	2; *H*; $55
5/day	**1; 53; $52**	**1; 55; $52**	**3; *H*; $52**	4; *H*; $53	5; *H*; $54	5; *H*; $55	5; *H*; $55
10/day	**1; 27; $52**	**1; 27; $52**	**1; 28; $52**	10; *H*; $53	11; *H*; $54	12; *H*; $55	13; *H*; $57
20/day	**1; 13; $52**	**1; 14; $52**	**1; 14; $52**	22; *H*; $53	24; *H*; $54	26; *H*; $56	27; *H*; $57
30/day	**1; 9; $52**	**1; 9; $52**	**1; 9; $52**	**35; *H*; $53**	39; *H*; $55	40; *H*; $56	41; *H*; $57
40/day	**1; 7; $52**	**1; 7; $52**	**1; 7; $52**	**48; *H*; $53**	53; *H*; $55	54; *H*; $56	56; *H*; $57
50/day	**1; 5; $52**	**1; 5; $52**	**1; 5; $52**	**62; *H*; $53**	66; *H*; $55	68; *H*; $56	71; *H*; $58
60/day	**1; 4; $52**	**1; 4; $52**	**1; 4; $52**	**75; *H*; $53**	80; *H*; $55	82; *H*; $56	86; *H*; $58

Table 9 The Optimal (q, T [hr], p [$]) for Various Parameter Values for $H = 3$ Days and $\bar{t}$ Derived from Proposition 2

	w						
λ	$0.1/day	$0.5/day	$1/day	$2/day	$3/day	$4/day	$5/day
0.5/day				No auction, $p = \$50$			
1/day				No auction, $p = \$50$			
2/day	**1; *H*; $52**	1; *H*; $53	1; *H*; $53	1; *H*; $53	1; *H*; $53	1; *H*; $53	1; *H*; $54
5/day	**1; 53; $52**	**1; 55; $52**	2; *H*; $52	2; *H*; $52	2; *H*; $53	2; *H*; $53	2; *H*; $53
10/day	**1; 27; $52**	**1; 27; $52**	**1; 28; $52**	4; *H*; $52	5; *H*; $53	5; *H*; $53	6; *H*; $54
20/day	**1; 13; $52**	**1; 14; $52**	**1; 14; $52**	11; *H*; $52	12; *H*; $53	13; *H*; $53	13; *H*; $54
30/day	**1; 9; $52**	**1; 9; $52**	**1; 9; $52**	**1; 9; $52**	19; *H*; $53	21; *H*; $54	21; *H*; $54
40/day	**1; 7; $52**	**1; 7; $52**	**1; 7; $52**	**1; 7; $52**	27; *H*; $53	29; *H*; $54	30; *H*; $55
50/day	**1; 5; $52**	**1; 5; $52**	**1; 5; $52**	**1; 5; $52**	35; *H*; $53	37; *H*; $54	39; *H*; $55
60/day	**1; 4; $51**	**1; 4; $52**	**1; 4; $52**	**1; 4; $52**	42; *H*; $53	45; *H*; $54	47; *H*; $55

Appendix 3. Results with More Knowledgeable Consumers

We assume that consumers evaluate the expected auction discount as if the total number of bidders of each type is given by the expected value of the relevant Poisson arrival process. We use this assumption to derive the suggested functional form for the threshold used by high-valuation consumers. We believe this assumption to be more realistic than to expect that consumers can accurately determine the expected discount from the auction using the Poisson distribution and Equation (5). To examine the effect of this assumption on our results, we look at the optimal design of the dual channel when consumers are so insightful that they evaluate the auction discount using the Poisson distribution of the number of bidders, that is, high-valuation consumers derive their participation threshold directly from Equation (5) by solving

$$\Pr(win \mid p)p - E[auction\_payment \mid p] = w\bar{t},$$

where

$$\begin{aligned}
&p\Pr(win \mid p) - E[auction_payment \mid p]\\
&\quad= p\Bigg(\sum_{x=0}^{q-1}\sum_{y=q-x}^{\infty}\frac{e^{-\lambda_1 T}(\lambda_1 T)^y}{y!}\frac{e^{-\lambda_2 T}(\lambda_2 T)^x}{x!}\\
&\qquad+\sum_{x=q}^{\infty}Q(x)\sum_{y=0}^{\infty}\frac{e^{-\lambda_1 T}(\lambda_1 T)^y}{y!}\frac{e^{-\lambda_2 T}(\lambda_2 T)^x}{x!}\\
&\qquad+\sum_{x=0}^{q-1}\sum_{y=0}^{q-x-1}\frac{e^{-\lambda_1 T}(\lambda_1 T)^y}{y!}\frac{e^{-\lambda_2 T}(\lambda_2 T)^x}{x!}\Bigg)\\
&\quad-\Bigg(\sum_{x=0}^{q-1}\sum_{y=q-x}^{\infty}\frac{e^{-\lambda_1 T}(\lambda_1 T)^y}{y!}\frac{e^{-\lambda_2 T}(\lambda_2 T)^x}{x!}O(q-x,y)\\
&\qquad+\sum_{x=q}^{\infty}pQ(x)\sum_{y=0}^{\infty}\frac{e^{-\lambda_1 T}(\lambda_1 T)^y}{y!}\frac{e^{-\lambda_2 T}(\lambda_2 T)^x}{x!}\\
&\qquad+R\sum_{x=0}^{q-1}\sum_{y=0}^{q-x-1}\frac{e^{-\lambda_1 T}(\lambda_1 T)^y}{y!}\frac{e^{-\lambda_2 T}(\lambda_2 T)^x}{x!}\Bigg)\\
&\quad=\sum_{x=0}^{q-1}\sum_{y=q-x}^{\infty}\frac{e^{-\lambda_1 T}(\lambda_1 T)^y}{y!}\frac{e^{-\lambda_2 T}(\lambda_2 T)^x}{x!}\\
&\qquad\cdot\left(p-\left(\underline{v}+(p-\underline{v})\frac{y-(q-x-1)}{y+1}\right)\right)\\
&\qquad+(p-R)\sum_{x=0}^{q-1}\sum_{y=0}^{q-x-1}\frac{e^{-\lambda_1 T}(\lambda_1 T)^y}{y!}\frac{e^{-\lambda_2 T}(\lambda_2 T)^x}{x!}.
\end{aligned}\tag{16}$$

Tables 10 to 12 present the optimal design of the dual channel for different values of H (the upper limit of the auction length). When consumers are as knowledgeable as the seller, the optimal auction length remains the longest possible, H time units, even for small w values. For one-unit auctions, the threshold found directly from Equation (5) is significantly larger than the threshold found based on Proposition 2. This means that on average, when consumers are knowledgeable, more high-valuation consumers will participate in the auction. As a result, most of the auction sales will be going to the class of buyers that the seller wants buying at the posted price. The only way the seller can increase revenue by adding auctions with small lots is to use long auctions to increase the expected auction price and deter the high-valuation consumers. For long auctions with large lots, the difference between these two thresholds (the threshold found based on Proposition 2, and the threshold found directly from Equation (5)) decreases as w and λ increase. Thus, for large values of w and λ, the two thresholds yield the same optimal design. For small values of w and λ, however, although short one-unit auctions perform very well in the first case (when the participation threshold is given by Proposition 2), they result in a decrease in revenue (compared to the posted-price channel alone) in the second case (when the participation threshold is derived from Equation (5)). Using the optimal designs listed in Tables 10–12, the seller can still increase his revenue by adding the auctions, but the increase is smaller than when consumers are less knowledgeable.

Table 10 **The Optimal** (q, T [hr], p [$]) **for Various Parameter Values for** $H = 7$ **Days and** $\bar{t}$ **Derived from Equation (5)**

	w						
λ	$0.1/day	$0.5/day	$1/day	$2/day	$3/day	$4/day	$5/day
0.5/day	No auction, $p = \$50$						
1/day	No auction, $p = \$50$		1; *H*; $52	1; *H*; $53	1; *H*; $53	1; *H*; $54	1; *H*; $54
5/day	4; *H*; $50	5; *H*; $51	6; *H*; $52	7; *H*; $54	8; *H*; $55	9; *H*; $57	9; *H*; $58
10/day	10; *H*; $50	13; *H*; $51	15; *H*; $52	16; *H*; $54	18; *H*; $56	19; *H*; $58	19; *H*; $59
20/day	25; *H*; $50	30; *H*; $51	32; *H*; $52	36; *H*; $54	38; *H*; $56	39; *H*; $58	40; *H*; $59
30/day	41; *H*; $50	47; *H*; $51	49; *H*; $52	56; *H*; $55	57; *H*; $56	59; *H*; $58	62; *H*; $60
40/day	55; *H*; $50	62; *H*; $51	68; *H*; $52	75; *H*; $55	79; *H*; $57	79; *H*; $58	83; *H*; $60
50/day	68; *H*; $50	80; *H*; $51	87; *H*; $52	93; *H*; $55	97; *H*; $56	100; *H*; $58	103; *H*; $60
60/day	81; *H*; $50	93; *H*; $51	101; *H*; $52	112; *H*; $55	117; *H*; $57	119; *H*; $58	125; *H*; $60

Table 11 **The Optimal** (q, T [hr], p [$]) **for Various Parameter Values for** $H = 5$ **Days and** $\bar{t}$ **Derived from Equation (5)**

	w						
λ	$0.1/day	$0.5/day	$1/day	$2/day	$3/day	$4/day	$5/day
1/day	No auction, $p = \$50$					1; *H*; $54	1; *H*; $54
5/day	2; *H*; $50	3; *H*; $51	3; *H*; $51	4; *H*; $52	5; *H*; $54	5; *H*; $55	5; *H*; $55
10/day	6; *H*; $50	8; *H*; $51	9; *H*; $51	10; *H*; $53	11; *H*; $54	12; *H*; $55	13; *H*; $57
20/day	16; *H*; $50	19; *H*; $51	22; *H*; $52	23; *H*; $53	25; *H*; $55	26; *H*; $56	27; *H*; $57
30/day	26; *H*; $50	32; *H*; $51	34; *H*; $52	37; *H*; $53	39; *H*; $55	40; *H*; $56	41; *H*; $57
40/day	37; *H*; $50	41; *H*; $51	46; *H*; $52	49; *H*; $53	53; *H*; $55	55; *H*; $56	56; *H*; $57
50/day	45; *H*; $50	54; *H*; $51	59; *H*; $52	63; *H*; $53	67; *H*; $55	68; *H*; $56	69; *H*; $57
60/day	58; *H*; $50	66; *H*; $51	72; *H*; $52	76; *H*; $53	80; *H*; $55	81; *H*; $56	85; *H*; $58

Table 12 **The Optimal** (q, T [hr], p [$]) **for Various Parameter Values for** $H = 3$ **Days and** $\bar{t}$ **Derived from Equation (5)**

	w						
λ	$0.1/day	$0.5/day	$1/day	$2/day	$3/day	$4/day	$5/day
1/day	No auction, $p = \$50$						
5/day	1; *H*; $50	1; *H*; $50	1; *H*; $51	2; *H*; $51	2; *H*; $52	2; *H*; $52	2; *H*; $53
10/day	2; *H*; $50	3; *H*; $50	4; *H*; $51	5; *H*; $52	5; *H*; $52	6; *H*; $53	6; *H*; $54
20/day	8; *H*; $50	9; *H*; $50	11; *H*; $51	12; *H*; $52	13; *H*; $53	13; *H*; $53	14; *H*; $54
30/day	13; *H*; $50	15; *H*; $50	17; *H*; $51	19; *H*; $52	20; *H*; $53	22; *H*; $54	22; *H*; $54
40/day	20; *H*; $50	23; *H*; $50	24; *H*; $51	27; *H*; $52	28; *H*; $53	29; *H*; $54	30; *H*; $55
50/day	24; *H*; $50	29; *H*; $50	32; *H*; $51	34; *H*; $52	36; *H*; $53	38; *H*; $54	39; *H*; $55
60/day	32; *H*; $50	35; *H*; $50	38; *H*; $51	40; *H*; $52	43; *H*; $53	46; *H*; $54	47; *H*; $55

Table 13 **The Optimal** (q, T [hr], p [$]) **for Various Parameter Values when the Two Channels Are Managed Independently**

	w						
λ	$0.1/day	$0.5/day	$1/day	$2/day	$3/day	$4/day	$5/day
1/day	3; *H*; $50	3; *H*; $50	3; *H*; $50	3; *H*; $50	3; *H*; $50	3; *H*; $50	3; *H*; $50
2/day	7; *H*; $50	7; *H*; $50	7; *H*; $50	7; *H*; $50	7; *H*; $50	7; *H*; $50	7; *H*; $50
5/day	17; *H*; $50	17; *H*; $50	17; *H*; $50	17; *H*; $50	17; *H*; $50	17; *H*; $50	17; *H*; $50
10/day	34; *H*; $50	34; *H*; $50	34; *H*; $50	34; *H*; $50	34; *H*; $50	34; *H*; $50	34; *H*; $50
20/day	69; *H*; $50	69; *H*; $50	69; *H*; $50	69; *H*; $50	69; *H*; $50	69; *H*; $50	69; *H*; $50
30/day	104; *H*; $50	104; *H*; $50	104; *H*; $50	104; *H*; $50	104; *H*; $50	104; *H*; $50	104; *H*; $50
40/day	139; *H*; $50	139; *H*; $50	139; *H*; $50	139; *H*; $50	139; *H*; $50	139; *H*; $50	139; *H*; $50

Appendix 4. Managing the Two Channels Independently

Table 13 shows the suboptimal channels' design that would result if the two channels were managed independently. The auction length and lot size are chosen to maximize revenue per unit time when all consumer have low valuations; i.e., there is no posted-price option. This is done by solving the following:

$$\max_{q,T}\left\{q\sum_{N=q+1}^{\infty}O\{q+1,N\}\frac{e^{-\lambda T}(\lambda T)^N}{N!}+R\sum_{N=0}^{q}N\frac{e^{-\lambda T}(\lambda T)^N}{N!}\right\}.$$

The result is that the auction design is independent of the waiting costs, and lot sizes tend to be larger than in Table 3. The posted price is set as if there is no auction channel: $\max(\underline{v}, \bar{v}/2)$, the solution to $\max \lambda \Pr(V \geq p)p = \lambda(\bar{v}-p)/(\bar{v}-\underline{v})p$. The result is a lower price than optimal, and too many high-valuation consumers purchase in the auction, leading to too few posted-price sales.

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