Problem set #2 answers

1. Three-firm Stackelberg: (a) Firm 2’s profit is $\pi_2 = (a - q_1 - q_2 - q_3 - c)q_2$, which leads to the best response $q_2 = (a - c - q_1 - q_3)/2$. Similarly, 3’s best response is $q_3 = (a - c - q_1 - q_2)/2$. The Nash equilibrium between 2 and 3 in the subgame following 1’s choice of $q_1$ then is $q_2 = q_3 = (a - c - q_1)/3$.

Anticipating this outcome, firm 1 maximizes $\pi_1 = [a - q_1 - q_2(q_1) - q_3(q_1) - c]q_1 = (a - c - q_1)q_1/3$. Maximization with respect to $q_1$ leads to the monopoly quantity, i.e. $q_1 = (a + c)/2$. This together with the strategies for 2 and 3 determined above is the unique SPE. The outcome is $q = (a-c/2, a-c/6, a-c/6)$.

(b) An example is the regular 3-firm Cournot equilibrium: $q_1 = (a - c)/4$, along with $q_2(q_1) = q_3(q_1) = (a - c)/4$ $\forall q_1$ (note that 2’s and 3’s strategies are functions, not just quantities!). Since it’s the Nash equilibrium in the static game, this must also be a NE in this game if 1 expects 2 and 3 to play $(a - c)/4$ no matter what 1 does. It’s not subgame-perfect, however, since in response to $q_1 \neq (a - c)/4$ the other two firms would not want to follow through with $(a - c)/4$ but instead choose the quantities specified in (a).

2. Pirates (a) Solve by backward induction. Suppose all plans are rejected until it is 317’s turn. Since only 317 and 318 are allowed to vote, 317’s plan can never be rejected with strict majority (clearly, 317 would not want to propose a plan that he would reject). Thus 317 allocates $1m for himself and $0 for 318, and this plan is implemented. Anticipating this outcome of the last subgame, 316’s optimal allocation is the allocation that maximizes his payoff subject to the constraint that the plan is not rejected. Thus, in addition to his own vote, he needs one more vote from either 317 or 318. It is optimal to give $1 to 318 (who would be made better off), $0 to 317 (who rejects 316’s plan anyway), and keep $1m-1 for himself. Anticipating this outcome, 315 chooses his optimal allocation. Since a strict majority is required for rejection, again only one vote in addition to his own is needed, which is most cheaply obtained by offering $1 to 317. 318 and 316 get $0, and 315 keeps $1m-1. And so on. In the SPE, each pirate votes against any plan that doesn’t offer at least $1 for himself. When proposing an allocation, every odd-(even-) numbered pirate offers $1 to all lower-ranked odd-(even-)numbered pirates, $0 to the even-(odd-)numbered pirates, and keeps the rest. The outcome is: The captain offers $0 to all even-numbered pirates, $1 to all other odd-numbered pirates and keeps $1m-158 for himself. The odd pirates
approve this plan, the even pirates reject, and hence the plan is implemented.

(b) One such equilibrium entails 2 through 318 rejecting any plan which gives less than $2 to each of them. Given this threat, 1’s best response is to give each $2 and keep the rest. Given this offer, and the anticipated continuation, the pirates accept, and we have an equilibrium, as the pirates never have to carry out their threat.

To see the problem with this equilibrium, suppose that there are only 4 pirates, and 2 through 4 threaten to reject any plan which offers them less than $2. What happens if 1 offers $0 to 2 and 4, and $1 to 3? 2 and 4 obviously reject, but 3? Looking forward, 3 can without much difficulty figure out that if the plan is rejected and 2 is next to propose a plan, he will get nothing. Thus it is better for 3 to accept 1’s plan.

Chances are, though, that if the captain proposes the plan determined in (a), it will be rejected, even if we assume that the pirates fully understand the game. In particular, if because of lack of foresight, fairness considerations, or just by “mistake” one of the pirates who are offered $1 rejects, the captain is out. This game is one of many examples showing that equilibria which are determined by iterated weak dominance in many round (which is what the backward induction procedure does) can lead to rather implausible outcomes, especially if some players clearly “lose” if they play according to the equilibrium (here, 2 through 318).

3. Sequential bargaining: This game is actually a bit simpler than the Rubinstein model in that any subgame starting in \( t = 1 \) (one after \( t = 0 \)) is identical to the whole game, not just the subgames starting in \( t = 2 \).

(a) For the same reasons as in the Rubinstein model, any SPE will entail agreement in \( t = 0 \), which means that a surplus of \( v \) is split between the players. Let \( p_1 \) denote the seller’s price in \( t = 0 \) if she is chosen to make an offer, and let \( p_2 \) denote the price offered by the buyer if he is chosen to make an offer. Let \( \pi \) denote the seller’s expected payoff in equilibrium, which means the buyer gets \( v - \pi \). I am taking a little short-cut here by assuming the equilibrium is unique. Actually establishing uniqueness follows the same strategy as for the standard Rubinstein model.

If 1 makes an offer, she chooses the largest \( p_1 \) that 2 will accept, given that in \( t = 1 \) 2’s expected payoff is \( v - \pi \). So we have \( v - p_1 = \delta(v - \pi) \), or \( p_1 = v - \delta(v - \pi) \). If 2 makes an offer, he chooses the smallest \( p_2 \) that 1 will accept, given that in \( t = 1 \) her expected payoff is \( v \). So we have \( p_2 = \delta v \). By definition, we have \( \pi = qp_1 + (1 - q)p_2 \). Plugging in the prices leads to \( \pi = q[v - \delta(v - \pi)] + (1 - q)\delta \pi \). Solving for \( \pi \) leads to \( \pi = qv \), and then we obtain \( p_1 = v - \delta(v - qv) \) and \( p_2 = \delta qv \). Thus, in the SPE, player 1 asks for \( p_1 \) and accepts 2’s price if it is at least \( \delta qv \). Player 2 offers \( p_2 \) and accepts 1’s price if it is at most \( p_1 \).

(b) Define 1’s expected equilibrium payoff as \( \pi \), which means that the payoff of each of the buyers is \( (v - \pi)/2 \). If 1 is chosen in \( t = 0 \) to make an offer, he must offer the buyer she is matched with at least \( \delta \frac{v - \pi}{2} \), so we have \( p_1 = v - \delta \frac{v - \pi}{2} \). If the buyer is
instead chosen to make an offer, the optimal price that the seller will accept is \( p_2 = \delta \pi \).

By definition, we have \( \pi = p_1/2 + p_2/2 \) and hence \( \pi = \frac{1}{2}(v - \delta^2 \frac{\pi}{2}) + \frac{1}{2} \delta \pi \). Solve for \( \pi \)
to obtain \( \pi = \frac{2 - \delta}{4 - 3\delta} v \). We then obtain \( p_1 = \frac{(2 - \delta)^2}{4 - 3\delta} v \) and \( p_2, p_3 = \frac{2 - \delta}{4 - 3\delta} v \). The complete SPE strategies take a form similar to those in (a). As \( \delta \) approaches 1, all players’ prices and the seller’s payoff converge to \( v \), while the buyers’ payoffs converge to 0.

The seller has a much more powerful position when bargaining with two buyers, since the seller can trade in every period, whereas each buyer matched with the seller knows that if he does not agree, the seller will be trade with the other buyer with probability 1/2 in the next period, and he will not trade. This asymmetry becomes more pronounced as players get more patient. As \( \delta \) converges to 1, the seller obtains all surplus.

4. Three-way bargaining:

For the usual reasons, in a SPE agreement will be reached in the first period (\( t = 0 \)), and the payoffs are \( (h, d, 1 - h - d) \). These must also be the (undiscounted) payoffs in \( t = 3 \) if the game were to get that far. In \( t = 2 \), Louie’s best allocation that the others will accept is \( (\delta h, \delta d, 1 - \delta h - \delta d) \). Thus, in \( t = 1 \), Dewey’s best allocation that the others will accept is \( \delta^2 h \) for Huey and \( \delta(1 - \delta h - \delta d) \) for Louie, which leaves (after simplifying) \( 1 - \delta + \delta^2 d \) for himself. In \( t = 0 \), therefore, Huey must offer Dewey \( \delta(1 - \delta + \delta^2 d) \), which by definition is \( d \) itself, and we obtain \( d = \frac{\delta(1 - \delta)}{1 - \delta^3} = \frac{\delta}{1 + \delta + \delta^2} \). Huey must offer Louie \( \delta^2(1 - \delta h - \delta d) \), which leaves \( h = 1 - \delta(1 - \delta + \delta^2) - \delta^2(1 - \delta h - \delta d) = 1 - \delta + \delta^3 h \).

Solving for \( h \) leads to \( h = \frac{1 - \delta}{1 - \delta^3} = \frac{1}{1 + \delta + \delta^2} \). Next, we have \( d = \delta(1 - \delta + \delta^2 d) \), which leads to \( d = \delta \frac{1 - \delta}{1 - \delta^3} = \frac{\delta}{1 + \delta + \delta^2} \). Finally, we can compute \( l \) either as \( l = 1 - h - d \) or as \( l = \delta^2(1 - \delta h - \delta d) \), both of which lead to \( l = \delta^2 \frac{1 - \delta}{1 - \delta^3} = \frac{\delta^2}{1 + \delta + \delta^2} \).

As indicated in the problem, one would obtain the same solution based on symmetry considerations: \( d = \delta h \), \( l = \delta d \), and \( h + d + l = 1 \); but that approach would leave unclear where subgame perfection comes in.

5. Gibbons, problem 2.11 Yes, such a SPE exists. First, notice that for 2, R is already a best response to B; so all we need to worry about in implementing (B,R) in the first period is a deviation by 1 to T. Second, the stage game has two pure-strategy Nash equilibria, (T,L) and (M,C), of which player 1 prefers the former. The candidate for a SPE is therefore: 1 plays B in \( t = 0 \). He plays T in \( t = 1 \) if (B,R) was played in \( t = 0 \), and M otherwise. Player 2 plays plays R in \( t = 0 \). She plays L in \( t = 1 \) if (B,R) was played in \( t = 0 \), and C otherwise. For any play in \( t = 0 \), the players’ actions in \( t = 1 \) are a Nash equilibrium of the subgame reached. In \( t = 0 \), player 2 is already playing a best response and can therefore not deviate to get a higher payoff in \( t = 0 \). She can also not gain from deviating in order to trigger a switch to the (M,C) equilibrium: her gain in \( t = 1 \) would be only 1, whereas the cost of deviating in \( t = 1 \) would be 2. Player 1, in turn has no incentive deviate since \( 4 + 3 \geq 5 + 1 \). So we have a SPE.