DYNAMIC COSTS AND MORAL HAZARD: A DUALITY BASED APPROACH

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Abstract

The marginal cost of effort often increases as effort is exerted. In a dynamic moral hazard setting, dynamically increasing costs create information asymmetry. This paper characterizes the optimal contract and helps explain the popular yet thus far puzzling use of non-linear incentives, for example in sales-force compensation. The result is obtained using two complementing dynamic programs – one based on duality and one similar to the standard. The dual program is monotonic and sub-modular, providing stronger results, including a proof for the sufficiency of one shot deviations.

Keywords: Dynamic moral hazard, nonlinear incentives, private information, dynamic mechanism design, duality, linear programming, stochastic programming, dynamic programming.

1. INTRODUCTION

Increasing marginal costs are a standard component of economic analysis. In organizational settings, the increase in cost often has a dynamic motivation. A worker picking fruits, for example, gets tired as the day progresses. In other settings the task itself becomes harder over time. Sales performance, for example, is measured over a period, typically quarter or year. As the quarter progresses, the agent depletes the “easy” sales leads and must exert more effort to generate later sales. Sales effort is inherently hard to monitor and pay is often performance based. If the firm knew the agent’s true cost, it would want to increase incentives towards the end of the quarter. This paper characterizes the optimal mechanism for a dynamic moral hazard setting in which the cost increase depends on the agent’s private information – his effort.

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Section 5.2 provides an example in which the optimal contract can be implemented using a known twist on a quota contract and a second example in which the optimal contract can be implemented by a slightly sophisticated convex incentive scheme. Joseph and Kalwani (1998) document the popularity of convex and quota based incentive schemes in sales related settings. However, as Prendergast (1999) summarizes: “rather remarkably, the theoretical literature has made little progress in understanding the observed (nonlinear) shape of compensation contracts, despite costs associated with nonlinearities.” The same conclusion is echoed in more recent studies, see e.g. Misra and Nair (2009) and Larkin (2007). Thus, the analysis here shows that increasing marginal cost can provide a relatively simple micro-economic foundation for the popularity of these schemes.

The model is the simplest possible to capture the problem of privately increasing costs. A risk neutral agent decides every day whether to exert costly effort. The probability of success (a sale) in the day increases with effort. The cost of effort today is a convex function of past effort.\textsuperscript{1} Effort is unobserved and the principal can commit to a contract at the outset.\textsuperscript{2}

To see the incentive problem, suppose that the probability of a sale each period is $\frac{1}{2}$ if the agent exerts effort and zero otherwise, and that the agent’s cost for making the $n$-th effort is $n$. If both the principal and the agent consider only current period incentives, a contract paying the agent $2n$ for a sale in day $n$ is incentive compatible and provides the agent zero expected utility – clearly first best. However, if the agent considers future payoffs, this contract is no longer incentive compatible. Shirking in the first period and then working whenever asked obtains the agent an expected utility of 1 each period. By shirking today the agent increases his rents from future work. The optimal contract must account for this additional incentive to shirk.

The optimal contract can be informally described as a dynamic quota: the agent starts in an evaluation stage and eventually moves to a compensation stage. In the compensation stage the agent is paid a fixed piece-rate for each sale and works for an additional fixed number of periods that is independent of any new outcomes. In

\textsuperscript{1}Multiple possible outcomes are considered in section B.1. The contract is essentially the same.
\textsuperscript{2}Abstracting from the imperfect commitment problem is a standard assumption in the dynamic moral hazard literature. See e.g., Rogerson (1985); Spear and Srivastava (1987); Fernandes and Phelan (2000); Clementi and Hopenhayn (2006); DeMarzo and Fishman (2007); Biais, Mariotti, Rochet, and Villeneuve (2010). For a recent examination of the implications of renegotiation in related settings see Strulovici (2011).
the evaluation stage the agent is rewarded *only* by changes to the expected fixed piece-rate, the length of the compensation stage, and the quota the agent must meet to enter the compensation stage. If the agent accumulates enough early successes, his compensation per sale later in the quarter will be high. If the agent did not accumulate enough early successes, the contract leads the agent to stop working.

Once the agent meets his “dynamic quota”, his reward is based only on his highest anticipated cost, generating excessive rewards for successful agents, as found in both Misra and Nair (2009) and Larkin (2007). On the other hand, the only way to profitably provide such high rewards is to limit the work by unsuccessful agents, resulting in a higher volatility of the work decision towards the end of the work period, consistent with the finding in Oyer (1998).

Related problems have been considered in the literature, simplifying various aspects of the problem. One natural simplification is to assume that the game is static. In static models the agent decides on the level of work before observing any outcomes. Innes (1990) show that a bonus for passing a certain threshold provides optimal profits and Kim (1997) shows that under certain conditions the outcome is first-best. The examples in section 5.2 confirm this. Poblete and Spulber (2012) restrict the piece rate of the contract to at most the revenue from the sale. In that case the optimal contract is a simple quota contract that pays the agent all the revenues above a certain thershold. The static contracts are generally sub-optimal and not incentive compatible in a dynamic setting as the agent has a richer deviation space.

If the agent’s marginal cost of effort fixed (i.e. non-increasing), the optimal dynamic contract can be derived using Clementi and Hopenhayn (2006). The contract will eventually either fire the agent without pay or allocate all remaining revenues to the agent. Thus, the optimal dynamic contract if costs are fixed is a form of a quota contract in which the threshold may change in response to early outcomes, but the reward for passing the threshold – all the remaining revenue – is independent of history. As a result, once the threshold is met, the continuation is first-best. Increasing

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3Biais, Mariotti, Rochet, and Villeneuve (2010) consider a closely related continuous time model with fixed marginal costs. There, the agent’s binary effort choice reduces the probability of a loss. Biais, Mariotti, Rochet, and Villeneuve (2010) however allow the principal to also choose the size of the project (roughly equivalent to multiplying both the revenue and cost to the agent in the current model). Biais, Mariotti, Rochet, and Villeneuve (2010) fully characterize the optimal contract in which the agent exerts effort at all times. In our setting the optimal contract for maximal effort up to any period \(T\) is a simple payment per success based on the period \(T\) cost. Instead, this paper focuses on characterizing the optimal contract.
marginal costs require that the reward for passing the threshold also depends on the history. In particular, the agent is unlikely to obtain all the remaining revenue and the continuation after passing the threshold is likely not first-best. This is because the first-best continuation is worth more to an agent that shirked because of the agent’s lower costs.

Increasing marginal costs are considered explicitly only in two period dynamic moral hazard problems. Mukoyama and Şahin (2005) is a direct application and Ábrahám, Koehne, and Pavoni (2011) extends the technical analysis to various two period settings. The economic analysis in Mukoyama and Şahin (2005) however focuses on the case that the agent’s effort in the first period complements effort in the second period. In such cases the agent’s compensation relies more on the second period outcome as second period incentives also induce effort in the first period. Consequently, the economic prediction in Mukoyama and Şahin (2005) is that the first period outcome affects the agent’s utility less than the second period outcome, and may in fact have no effect. In contrast, increasing marginal costs imply that the agent’s efforts are substitutes and thus the prediction here is opposite.

For more periods, Fernandes and Phelan (2000) show that the one-shot-deviation (OSD) condition is violated. That is, there are contracts in which the agent has a profitable multi-period deviation but no profitable single deviation. Because the standard recursive methods assume the OSD condition holds, they cannot be used to characterize the optimal incentive compatible contract. To alleviate this problem, Mukoyama and Şahin (2005)\(^4\) consider a long term relationship in which the agent’s actions only affect two periods (the current and the next). DeMarzo and Sannikov (2008) study a setting that is very similar to the one studied here, but consider only the aggregation problem (as in Holmstrom and Milgrom (1987)). The additional aggregation problem changes the analysis and results.\(^5\)

An earlier approach to dealing with changing costs, dating back to Berliner (1957) and Weitzman (1980) is to consider any attempt to adjust the optimal contract to the increasing difficulty as a form of ratcheting which should not be profitable. The intuitive argument is that the agent will always have a profitable way to ‘game the

\(^4\)see also Tchistyi (2006).

\(^5\)Other models either focus on dynamic persistence of private information that is not controlled by the agent (Williams (2011)) or on settings in which the surplus depends on the private history but the production ends after the first success (see Bergemann and Hege (2005), Bonatti and Hörner (2011) and Halac, Kartik, and Liu (2012)).
system’. This intuition, echoed in the cited empirical literature, implies that linear contracts should be optimal. Our analysis shows that this intuition is only half right. If we limit attention to contracts that require the same amount of total effort regardless of early outcomes, as these early papers (implicitly) did, there are, indeed, no gains from ratcheting. However, if we allow contracts to also adjust the required effort, as modern dynamic contracts do, then ‘ratcheting’ not only increases the principal’s expected profit, but in some cases (see the second example of section 5.2) increases also the agent’s expected profit and thus total surplus.

Relative to this literature, the current paper’s contribution is a characterization of the optimal contract for settings with increasing marginal cost. Allowing this richer production setting results in contracts that are consistent with empirical observations. In particular, the contracts in the existing literature eventually provide the agent a unique piece-rate that generates the first best continuation. That is, a contract in which the agent may receive a 5% comission in some realizations and a 15% comission in another is difficult to explain using the existing static or dynamic contracts.\footnote{Allowing for private information on the agent’s side of his initial productivity introduces a menu of such contracts, while here the single contract offers several comission levels.}

In practice, incentive contracts frequently do have several commission levels (see e.g. Joseph and Kalwani (1998) and Larkin (2007)). Moreover, the existing dynamic contracts all predict that once the agent is paid, the ex-post continuation is first best. While the prediction is hard to test directly, it is difficult to reconcile with the observed convex incentive schemes.

To confront the challenge of persistent private information, this paper introduces a reformulation of the dynamic contract problem that is based on duality. This reformulation provides additional characterization of the optimal contract. In particular, it proves that while some IC contracts may violate OSD, the optimal contract does not.

To understand the dynamic dual intuition, suppose that before signing the contract, the principal and agent learn that some third party will have the power to stop the contract at history $h$. The principal must decide, at this early stage, how much to spend now to prevent this termination. The dual value of history $h$ is the largest number that the principal would pay.

To determine this dual value, we must consider (a) the expected continuation revenue and (b) the expected cost of incentivizing the agent to exert the required continuation...
effort. However, these do not capture all the implications. If everyone knows ex-ante that the contract will terminate at history $h$, the utility provided to the agent starting in $h$ from the optimal contract can no longer provide incentives (or disincentives) for effort. For example, in most contracts, the agent is not paid for success in the first period. Instead, he is rewarded by a better contract following success than following failure. If the agent knows that the third party will terminate the contract after the first success, giving him zero continuation utility, he may well require payment for the first success. Thus, the dual value must also consider \textit{(c) the effect of the agent’s continuation utility on previous histories.}

The three considerations above are common to all dynamic moral hazard settings. The current setting adds \textit{(d) the effect of the agent’s continuation cost gain on previous history.} That is, if the contract terminates, any private information gains to the agent from shirking in the past to obtain a lower cost today are destroyed. If the agent knows this ex-ante, his incentives to shirk are weaker, reducing the principal’s costs.

The total dual value of the history accounts for all four elements (a)–(d) above. In contrast, suppose the third party termination threat is completely unexpected by both parties and just appears in history $h$. In this case the principal would pay the up to her expected continuation profit from history $h$. The value of avoiding this unexpected termination is given by the standard dynamic moral hazard value (cf. Spear and Srivastava (1987)). The difference is illustrated best by proposition 4, which shows that the standard value may be negative especially when dual value is all the remaining surplus.

To determine the dual value, the standard ‘promised utility’ state variable (cf. Spear and Srivastava (1987)) is transformed to the \textit{marginal cost to the contract of the agent’s utility}. This exactly captures (c) above. In addition, a state variable indicating the \textit{marginal cost to the contract of the agent’s private information} is used to capture (d) above. Both of these costs are derived by aggregating the shadow prices (multipliers) on the incentive constraints in the preceding histories. Intuitively, the agent’s utility in history $h$ either relaxes or constraints each of the preceding ICs. For a history $h′$ that preceded $h$, the IC’s shadow price (along with the coefficients in the IC itself) determines the marginal cost to the contract of changing the utility in $h$. Aggregating the shadow price over all the histories that precede $h$ provides the required state variable for (c). The same intuition provides the state variable for (d).

To prove it is sufficient to consider only OSD, the dynamic dual analysis considers
a change that increases the expected profits relative to the optimal contract. This change must violate some incentive constraint, otherwise the original contract would not be optimal. If the constraint that is violated is always a one-shot-deviation constraint, then the optimal contract subject only to OSD must be IC.

As the dual state variables reflect costs, the dual value is monotonic in each state variable. In addition, in our setting the costs – and corresponding state variables – are substitutes. This allows proving stronger results than typical for the optimal contract, and in particular the OSD result.

Duality based approaches are widely used in economic modeling, dating back to Rockafellar (1970). Vohra (2011) extends the analysis of static adverse selection models by analyzing the dual of the classic adverse selection problem. Marcet and Marimon (2011) and Mele (2011) consider shadow multipliers in a dynamic setting that can be applied to moral hazard problems. The formal analysis however assumes the OSD assumption holds. Abraham and Pavoni (2008) use a mixture of a shadow variable and the promised utility to construct a recursive model of savings and consumption. Their approach however relies on a numerical procedure to verify ex-post that the first order approach is valid.

The methodological contribution of the paper is a recursive dual formulation that is intuitive, tractable, captures additional frictions (the information rent) and can be extended to other settings. In particular, settings without a proof for sufficiency of OSD.

Section 2 lays out the dynamic production model. Section 3 provides the dual form and its economic interpretation. The optimal contract is characterized in section 4. Section 5 considers some specific extensions and illustrative examples. Section 6 concludes.

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7Standard dynamic moral hazard analysis uses the agent’s continuation utility as the state. If the agent’s continuation utility is exactly his outside option, the contract must typically terminate and the principal obtains his outside option. If the agent’s continuation utility is very high, the principal must either give away the firm (in limited liability), or provide the agent costly insurance. In all cases, the principal’s expected continuation value is highest for some expected agent’s continuation utility between the two extremes and is thus non-monotonic. One way around this (with its advantages and disadvantages) is to reformulate the problem as maximizing the total surplus (see e.g. Clementi and Hopenhayn (2006)).
2. MODEL

2.1. Setup and Primitives

There is a principal and an agent, both risk neutral. Both have an outside option set to zero. The agent has limited liability – i.e. money can only be transferred to the agent. Time is discrete. In each period the agent either works or not. The agent’s work is costly to the agent and unobservable to the principal. The cost of effort in a period is $c_n$ for a commonly known function $c : N \rightarrow R_+$ where $n$ denotes the number of actual periods of work. That is, for the first period of work, $n$ is one. For the second period of work, $n$ is two, and so on. If the agent shirks in the first period, $n$ in the second period is still one. The analysis will focus on the case that $c_n$ is an increasing and convex function. However, the dual methodology that is developed does not depend on any of the assumptions on $c$.

**Assumption 1** The agent’s cost, $c_n$, is increasing and convex in the work period number, $n$.

Both the principal and the agent observe the outcome of each period. To simplify the exposition, a period’s production outcome is either success or failure, denoted by $y \in Y = \{0, 1\}$. Appendix B.1 shows the results extend directly to any countable set of possible outcomes. The principal earns a revenue of $v$ from each success ($y = 1$) and zero from a failure ($y = 0$). The probability of success (resp. failure) in a period in which the agent works is $p \in (0, 1)$ (resp. $1 - p$). If the agent does not work, $p$ is replaced with $p_0 \in [0, p)$. To prevent the principal from making free profits, assume the principal incurs a cost of $v \cdot p_0$ for every period in which the contract is still active.

As costs are increasing, the surplus from working becomes negative after enough effort was exerted. Let $N^{FB}$ denote the maximum number of periods in which consecutive work increases surplus:

$$N^{FB} = \max n : c_n \leq v (p - p_0) .$$

The increase in costs implies that an infinite contract is never optimal. The exposition is simplified by assuming that the agent and principal do not discount the future.

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8This assumption is common to dynamic moral hazard models. Allowing the agent to deviate by shifting one period’s outcome to a later generates additional interesting questions also for the standard models. These remain for future research.

9This assumption only simplifies the exposition and is without loss of generality.
Appendix B.2 shows that adding a discount factor amounts to a simple accounting exercise.

The following observation simplifies the exposition and notation. All formal results without proof in the text are proved in the appendix.

**Lemma 1**  
*All contracts that are incentive compatible satisfy individual rationality. There is an optimal contract in which:*

1. The agent works for at most $N_{FB}$ periods
2. The required work decision is a stopping decision: if the agent is ever asked not to work, the contract terminates.
3. The agent is never paid in a period with a failure or without required work.

The first two are standard and the last is a direct outcome of risk neutrality. Note that the second statement does not rule out randomization of the stopping decision.

Given lemma 1, the space of contract relevant public histories $H$ is the space of previous outcomes:

$$H = \bigcup_{n=0}^{N_{FB}} Y^n.$$  

A public history $h \in H$ denotes a sequence of outcomes. Part two of lemma 1 implies that if the contract is not yet terminated, the agent was asked to work in all past periods. However, only the agent knows in which periods he actually did work and in which periods he shirked. As the cost to the agent of working in a period is a function of the number of periods in which the agent actually worked in the past, the only information in the agent’s private history that is payoff relevant is the number of past shirks:

**Definition 1** The agent’s *private* history $(h, s)$ is the public history $h$ and the number of past shirks $s$.

Let $n_h$ denote the number of the period just after history $h$. If the agent did not deviate in the past, his cost of work in this period will be $c_{n_h}$. However, cost depends on the *private* history. With a slight abuse of notation let $c_{h-s} \equiv c_{n_h-s}$ denote the cost for any history $h$ with past deviations $s$ and $c_h \equiv c_{h-0}$. As the difference in cost between two work periods will play an important role, let $d_{h-s} \equiv c_{n_h-s} - c_{n_h-s-1}$ denote the
cost difference between the current and previous periods if the agent shirked \( s \) times in the past and \( d_h = d_{h-0} \). To simplify the notation later on, set \( d_1 = c_1 \).

The analysis makes extensive use of histories following and preceding other histories. Let \( h = \langle h^1, h^2 \rangle \) denote the history \( h^1 \) followed by the history \( h^2 \). That is, the sequence of outcomes \( h^1 \) happened and then the sequence \( h^2 \) happened. For example, if the current history is \( h \), then the next history will be either \( \langle h, 1 \rangle \) or \( \langle h, 0 \rangle \). Say that the history \( \langle h^1, h^2 \rangle \) follows history \( h^1 \) and denote the “follows” relation by \( \succeq \). That is:

\[
\tilde{h} \succeq h \iff \exists \hat{h} \in H :\, \tilde{h} = \langle h, \hat{h} \rangle.
\]

As the set \( H \) includes the empty set, \( \hat{h} \succeq h \).

### 2.2. The Contract

There is no loss in considering only contracts that specify for each period a work decision and a wage based on the period’s public history. This section defines the contract using a simple transformation of the common decision variables. This transformation will allow a linear formulation of the problem without affecting the interpretation of the resulting contract.

Typically, the contract specifies for each history whether the contract terminates in the corresponding period and the resulting wage. Let \( (1 - a_h) \) denote the probability that the contract is terminated in history \( h \), if \( h \) is reached. Let \( W_h \) denote the wage paid for success in the period.

The ex-ante probability that the contract would still be active after a success is \( a_\emptyset \cdot a_{\{1\}} \). To linearize the formulation, the contract will use the ex-ante probability that the principal did not decide to terminate the contract yet instead. This probability, denoted \( q_h \), can be defined recursively from any \( a \) and vice versa:

\[
q_\emptyset = a_\emptyset \quad \text{and} \quad q_{\langle h, y \rangle} = q_h \cdot a_{\langle h, y \rangle} \tag{2.1}
\]

The payment \( W_h \) is only paid if the contract was not terminated by period \( h \). Thus, the expected for success in a history is \( W_h \cdot q_h \), which is again non-linear. However, by lemma 1, there is no loss of generality in having the contract specify the wage for success in the history, conditional on the contract not terminating yet.\(^{10}\)

\[
w_h \equiv q_h \cdot W_h \tag{2.2}
\]

**Definition 2** A contract is a pair of functions \( \langle q, w \rangle \), with \( q_h \) specifying for each

\(^{10}\)See Abreu, Milgrom, and Pearce (1991) for a similar linearization.
history $h$ the cumulative probability that the agent will be still be asked to work in the period (equations 2.1); and $w_h$ specifying the wage for success, weighted ex-ante by the probability $q_h$ (equation 2.2).

**Remark 1** As the agent and principal are risk neutral, there are infinitely many equivalent ways for the optimal contract to pay the agent $w$ dollars. Paying a dollar for success today is equivalent to paying $\frac{1}{p}$ more dollars for success tomorrow (assuming the agent is asked to work). To remove this technical duplication of the optimal contracts, the analysis assumes the optimal contract makes payments “as early as possible.” That is, from all contracts that specify the same work plan, the optimal contract is the one that pays most to the agent as early as possible.

If the agent complies with the contract (i.e. works when asked to), the expected continuation profit for the principal starting from a history $h$ is given by:

$$V^h = q_h (p - p_0) v - pw_h + pV^{(h,1)} + (1 - p) V^{(h,0)}.$$  

(2.3)

The only deviation from the standard formulation is that $q_h$ and $w_h$ also include the probability that the contract was terminated at any history before $h$. The sum $(p - p_0) v$ is the expected revenue from work, the agent’s expected payment is $p \cdot w_h$ and the last two terms are the continuation value.

A similar expected value can be defined for the agent. Letting $U^h$ be the agent’s expected continuation utility from complying with the contract in all remaining periods if he never shirked in the past, we have:

$$U^h = pw_h - q_h c_h + pU^{(h,1)} + (1 - p) U^{(h,0)}.$$  

However, if the agent did shirk in the past, his expected profit increases. The approach introduced by Fernandes and Phelan (2000) suggests defining the agent’s expected utility conditional on $s$ past deviations. We adopt a slightly different approach, identifying separately the agent’s extra gains from previous shirks.

In particular, let $D^h_s$ denote the *increase* in the agent’s expected utility starting in private history $(h, s)$ if he would have made one more shirk at some point in the past. That is $D^h_0$ is the agent’s extra gains starting at public history $h$ if he would have made exactly one shirk in the past, $D^h_1$ is the agent’s gain from making a second shirk in the past, and so on. As the additional gains are only a result of the cost difference, we have that:

$$D^h_s = q_h d_{n-s} + pD^{(h,1)}_s + (1 - p) D^{(h,0)}_s.$$
The agent’s expected utility conditional on s past deviations, denoted $U^h_s$ is:

$$U^h_{s+1} = U^h_s + D^h_s \quad \text{with} \quad U^h_0 = U^h.$$ 

The optimal contract chooses $q, w$ that maximize $V^\emptyset$ subject to incentive compatibility (IC) and individual rationality (IR). As the agent can always choose to stop working IR is implied by IC and thus will be subsequently ignored.

### 2.3. Incentive Compatibility

To construct the IC constraints, first assume that the OSD condition holds. That is, it is necessary and sufficient that the agent does not have a profitable single deviation. As in related treatments, I will refer to the relevant constraints to prevent one-shot deviations as Local Deviation Incentive Constraints (LDIC).

The LDIC for history $h$ requires that whenever the agent considers a one and only deviation, his expected continuation utility is lower than from complying with the contract in all future periods ($U^h$). If the agent deviates in history $h$, his expected payment in the period is $p_0 w_h$. The agent expects to succeed and transition to the history $\langle h, 1 \rangle$ with probability $p_0$. As the agent is considering a one-and-only deviation, his expected continuation utility after shirking and succeeding is $U^\langle h, 1 \rangle + D^\langle h, 1 \rangle_0$. Similarly, with probability $(1 - p_0)$ the agent expects the continuation utility $U^\langle h, 0 \rangle + D^\langle h, 0 \rangle_0$.

The LDIC is therefore

$$(LDIC) \quad pw_h - q_h c_h + pU^\langle h, 1 \rangle + (1 - p) U^\langle h, 0 \rangle \geq p_0 w_h + p_0 \left( U^\langle h, 1 \rangle + D^\langle h, 1 \rangle_0 \right) + (1 - p_0) \left( U^\langle h, 0 \rangle + D^\langle h, 0 \rangle_0 \right)$$

It is clear that IC implies LDIC. However, as in Fernandes and Phelan (2000), in the current model, LDIC does not imply IC.

**Lemma 2** A contract may satisfy all LDIC but violate IC.

The proof constructs a contract in which all LDIC hold with equality while the agent’s optimal plan is to shirk in the first two periods.

LDIC is not sufficient because some contracts may require more work in some histories that follow a failure than other histories that follow success. As a result, the private information (lower costs) gains from shirking and failing may be too large. To prove that the optimal contract subject to LDIC is indeed IC, consider another set,
the “Final Deviation Incentive Constraints” (FDIC). The FDIC require that whenever an agent who possibly shirked in the past considers one final shirk, he prefers to follow the contract in all later periods.

Applying the same logic as for the LDIC, obtains the following formulation of the FDIC for private history \((h, s)\):

\[
\text{(FDIC)} \quad pw_h - q_h c_{h-s} + p \left( U^{(h,1)} + \sum_{j=0}^{s-1} D^{(h,1)}_j \right) + (1 - p) \left( U^{(h,0)} + \sum_{j=0}^{s-1} D^{(h,0)}_j \right) \\
(2.5) \quad \geq \quad p_0 w_h + p_0 \left( U^{(h,1)} + \sum_{j=0}^{s} D^{(h,1)}_j \right) + (1 - p_0) \left( U^{(h,0)} + \sum_{j=0}^{s} D^{(h,0)}_j \right)
\]

The FDIC 2.5 reflects the work/shirk tradeoff for an agent that already shirked \(s\) times and is considering a final shirk. Note that the sums go only to \(s - 1\) if the agent does not shirk (the top line) and to \(s\) if the agent does shirk (the bottom line). The next lemma verifies the relation between FDIC and IC:

**Lemma 3** If a contract is FDIC it is IC

The intuition for the lemma is simply that any profitable deviation plan must have a profitable last deviation. However, FDIC is a stricter condition than IC. For example, it may be that the first deviation was more costly to the agent than the gain from the second deviation. As FDIC is stricter than IC which in turn is stricter than LDIC, the following corollary follows:

**Corollary 1** If every optimal contract subject to LDIC satisfies FDIC, then every optimal contract subject to LDIC is optimal subject to IC.

The proof for sufficiency of LDIC will show that the condition of corollary 1 does in fact hold. For this, we must analyze the FDIC problem.

2.4. The FDIC Problem

Problem 2.6 is the dynamic FDIC problem. To save on notation, \(\vec{D}\) stands for the vector of \(D\)’s for all relevant private history \(s\) values and a superscript \(y\) denotes the
possible outcomes. The shadow cost for each constraint is provided as well:

\[
V (n, \bar{q}, U, \bar{D}) = \max_{(U^s, \bar{D}^v, q, w) \geq 0} q (p - p_0) v - p \cdot w
\]

\[
+ p V (n + 1, q, U^1, \bar{D}^1) + (1 - p) V (n + 1, q, U^0, \bar{D}^0)
\]

subject to

\[
\text{Probability (}\mu\text{)} \quad q \leq \bar{q}
\]

\[
\text{FDIC } \forall s \leq n - 1 \quad (\lambda_s) \quad \text{Constraint 2.5}
\]

\[
\text{Regeneration } U : \quad (\gamma) \quad U = pw - qc + pU^1 + (1 - p) U^0
\]

\[
\text{Regeneration } -D \forall s \leq n - 1 \quad (\delta_s) \quad D_s = qd_{n-s} + pD_s^1 + (1 - p) D_s^0
\]

The optimal contract solves

\[
\max_{V \geq 0, \bar{D} \geq 0} V (1, 1, U, D)
\]

There are three non-standard elements in problem 2.6. First, the variable \(q\) and the accompanying probability constraint. In the standard notation, the upper bound on the probability of work is '1'. Here, as \(q_h\) identifies the ex-ante probability, the upper bound is simply the previous period’s probability, which may be lower than 1:

\[
0 \leq q_0 \leq 1 \quad , \quad 0 \leq q_{(h,y)} \leq q_h
\]

The probability constraint, together with the determination of the next period’s \(q\) enforces this change.

The second non-standard element are the FDIC for each private history \(s\), instead of just the one LDIC per history.

Finally, the ‘regeneration constraints’ for the shirking gains (\(D\)) are added. These are called the ‘threat-keeping constraints’ in Fernandes and Phelan (2000). The derivation there details the need for these for \(s = 1\) assuming the one-shot-deviation principle holds. If OSD does not hold, the problem must have these for all possible private histories, as is the case here. The intuition for the additional regeneration constraints is simple. As the problem must set some continuation values after deviation in the FDIC, the dynamic problem must recursively define those values and maintain these just as it does for the original utility in the standard formulation.

The Dynamic LDIC problem is the same as 2.6 with the FDIC only for \(s = 0\) and
the D regeneration constraint only for \( s = 0 \).

There are several difficulties with the dynamic problem. First, it is not well defined for all values of \( U \) and \( \vec{D} \). For example, there is no solution in which \( U = 0 \) and \( D_0 > 0 \). This complicates proving even “standard” results, such as concavity. Second, one cannot prove using this problem that in the optimal LDIC contract all FDIC are slack, which is critical for the remainder of the analysis. One sub-optimal contract that satisfies LDIC but not FDIC is illustrated in appendix A.2.

Formally, the standard line of proof would show that optimality (or first order conditions) imply that the LHS of the FDIC for \( s + 1 \) is larger than the LHS of the FDIC for \( s \). Simplifying, this is

\[
d_{n-s} + p \left( D_s^1 - D_{s-1}^1 \right) + (1 - p) \left( D_s^0 - D_{s-1}^0 \right) \geq p_0 \left( D_{s+1}^1 - D_s^1 \right) + (1 - p_0) \left( D_{s+1}^0 - D_s^0 \right)
\]

The technical challenge here turns out to be that the \( D_{s+1}^0 - D_s^0 \) term may violate the inequality if the agent works more after failure than after success, as \( 1 - p_0 \) is larger than \( 1 - p \).\(^{11}\)

While the dual analysis below can be used to show that this condition holds in the optimal contract, it will provide a more direct proof for sufficiency of the LDIC.

3. DUAL FORM

The characterization of the optimal contract in section 4 relies heavily on a dual formulation of the problem that is developed in this section. Proposition 1 establishes the dynamic dual representation of problem 2.6 is problem 3.5. The exposition here uses first order conditions and partial derivatives for the formal arguments. This makes explicit the underlying economics and the (limited) importance of the linearity of the problem. For simplicity, the exposition will focus on the LDIC problem when indicating formal results for history \( h \). Once this is understood, the FDIC dual is a technical generalization. A formal derivation using linear programming duality that does not assume differentiability is provided in the appendix. If differentiability and concavity holds, the Slater Condition (see e.g. Borwein (2005)) can be used instead. In this case, all the development in the text applies directly as a proof for proposition 1.

Intuitively, consider the optimal contract problem at history \( h \). The continuation

\(^{11}\)A separate line of proof can be used uniquely for the case that \( p_0 = 0 \).
contract affects the agent’s incentives, and the principal’s profits for the periods that precede \( h \) as well as those that follow it. However, the standard (primal) recursive value accounts directly only for the continuation profit. The dual value accounts for the costs and benefits a continuation imposes on all periods – those that precede it as well as those that follow it.

The next subsection develops the preceding histories cost intuition in detail. To achieve this, the dual analysis leverages two insights that apply to any optimal contract:

1. In any optimal contract, the cost and benefit imposed by any continuation \( h' \) on history \( h \) that precedes it can be inferred from history \( h \)'s ICs and their shadow prices.
2. In any optimal contract, the continuation benefit (or cost) from changing a state variable (e.g. the agent’s promised utility) must equal the cost imposed on the preceding periods.

Finally, with the complete effect of the optimal continuation for each history at hand, the standard dual intuition can be applied: The optimal dual value is determined by starting from the ‘best case’ for the principal (i.e. “agent works for free”) and accounting for the various constraints and costs.

The conclusion is the dynamic dual problem, provided in proposition 1.

3.1. Effect on Preceding Histories

Dynamic moral hazard analysis finds the best continuation given some fixed past that is encoded into a state variable. Since Spear and Srivastava (1987), the past is typically encoded into the agent’s promised utility. In the dynamic dual analysis, we replace the agent’s utility as a state variable with the optimal cost to provide that utility. This state variable allows us to determine directly the optimal continuation utility in history \( h \) instead of solving for the optimal continuation given any utility. It is convenient to consider this preceding-histories effect as a cost rather than a benefit, this is of course a cosmetic choice and has no formal implication. In particular, the cost may be negative (i.e. a benefit).

Consider a marginal increase to the agent’s utility in the second period after failure in the first (\( U^0 \)). Suppose this increases the continuation profit in the second period by a small \( \varepsilon \). Should it be done? To answer this question, we need to also determine the effect on the first period profit. Additional utility after failure decreases the agent’s
incentive to work. In particular, if the IC in the first period was binding, additional utility after failure implies that the IC is now violated. For convenience, the LDIC 2.4 for a general history $h$ is repeated here (omitting the $h$ for simplicity):

\[(LDIC) \quad pw - qc + pU^1 + (1 - p) U^0 \geq p_0 w + p_0 (U^1 + D^1_0) + (1 - p_0) (U^0 + D^0_0)\]

A marginal increase in $U^0$ violates the LDIC at a rate of $(p - p_0)$. Because the agent is indifferent between utility and the payment for success in the first period, the increase in $U^0$ can be compensated by increasing the agent’s pay in the first period at the same rate. However, it may also be compensated by increasing the utility after success $U^1$. The increase in $U^1$ may increase continuation profit and thus be more profitable. It therefore seems as if to determine the correct marginal cost of increasing $U^0$ we must know the optimal contract after success as well.

The first key insight is that in any optimal contract, the correct formalization of the effect of a change in any of the continuation variables is captured by the shadow cost of that period’s IC and only by it. All that is required to determine the effect of a change in second period utility on first period profit is the first period IC (or ICs) in which the second period variable appears and the correct shadow costs for these ICs. In particular, if $\lambda$ is the correct shadow cost of LDIC 3.1, an increase in $U^0$ decreases the optimal value in the first period by $\lambda \cdot (p - p_0)$ by construction, regardless of the specific continuation.

We can now return to the first question: if a marginal increase in $U^0$ increases the continuation profit starting at $U^0$ by $\varepsilon$ and decreases the preceding histories profit by $\lambda (p - p_0)$ it should be done if and only if

\[(3.2) \quad \lambda (p - p_0) \leq (1 - p) \varepsilon .\]

The LHS of 3.2 is the cost implied on the preceding history. The RHS is the change in profit, accounting for the probability of getting to history $U^0$.

In the optimal contract, the inequality that is equivalent to 3.2 should never be slack in any history. If it is, the contract is improved by increasing the agent’s promised utility. Similar logic implies that if the inequality is ever violated, the optimal contract is improved by decreasing the agent’s promised utility – the cost saving on preceding histories is larger than the loss of continuation profit.
This is the second key insight: in any optimal contract, the marginal continuation benefit of increasing the agent’s utility must exactly equal the marginal cost implied on the preceding histories. Otherwise, if the former is larger, the optimal contract should promise the agent more utility. If the latter is larger, the optimal contract should promise the agent less.

Thus, optimality implies two insights: cost on preceding histories is captured by the IC; and the net effect of any marginal change must be zero when accounting for both the continuation and preceding histories. To formalize these insights, observe first that applying the envelope theorem to problem 2.6, using the shadow variable $\gamma$ implies that\footnote{Note again that differentiability is not required for the formal derivation in the appendix but is assumed throughout the exposition for simplification only. Nevertheless, if the original dynamic problem is differentiable the development here in the exposition provides a complete proof of proposition 1, subject to the Slater condition (or another duality restriction).}

\begin{align}
(3.3) \quad \forall h : \quad \frac{dV}{dU} = \gamma
\end{align}

in any optimal contract. That is, the effect of the agent’s utility on continuation profit is $\gamma$. The second insight above implies that in the optimal contract, this is also the marginal cost on the preceding periods of providing the agent more utility. Again assuming differentiability, and focusing only on the LDIC problem, the first order conditions for $U^1$ and $U^0$ formalize this:

\begin{align*}
 p \cdot \frac{dV^1}{dU^1} + \lambda (p - p_0) - p \cdot \gamma &= 0 \\
 (1 - p) \frac{dV^0}{dU^0} - \lambda (p - p_0) - (1 - p) \gamma &= 0
\end{align*}

The continuation effect (the first term in each), must equal the effect on the current period (the second term) and any effect on the previous periods (which by recursion is captured exactly by $\gamma$). In the very first period, there are no preceding histories and $\gamma = 0$. This is equivalent to the standard static condition that the foc is zero. Observe that indeed with the third term exactly zero the equation for $U^0$ in a two period setting is exactly the same as 3.2.

The envelope theorem (equation 3.3) implies that it is correct to replace $\frac{dV^y}{dU^y} = \gamma^y$.

This gives rise to a law of motion for the shadow cost of the agent’s utility:

\begin{align*}
 \gamma^1 = \gamma - \frac{\lambda (p - p_0)}{p} \quad \text{and} \quad \gamma^0 = \gamma + \frac{\lambda (p - p_0)}{1 - p} .
\end{align*}
In any optimal contract, in any history $h$, the marginal cost on all the preceding histories of providing the agent utility in a history must be $\gamma^h$. This cost is derived by aggregating all the shadow costs in all the ICs for the histories that preceded $h$, as indicated recursively in the two laws of motion above. Optimality implies that in any period, $\gamma^h$ is also the shadow cost on the regeneration constraint for promised utility.

The analysis so far considered only the continuation utility, which is common to our model and all existing models of dynamic moral hazard. Our problem has another type of continuation variables – the extra ‘private’ gains from shirking $D^y_0$ ($D^y_s$ in the FDIC). The same logic applies for these as for the continuation utility:

- Use the IC in all the preceding histories to evaluate the preceding-history-cost of changing the agent’s private gains in a period ($D$). In particular, the contribution of each to the IC in every period is additively separable.
- Observe that in the optimal contract, any continuation gains from increasing $D$ must exactly equal the cost implied on preceding histories.
- By the envelope theorem

$$\frac{dV^h}{dD^h} = \delta^h$$

- Applying the envelope theorem to the first order condition recursively obtains the laws-of-motion:

$$\delta^1 = \delta + \lambda \frac{p_0}{p} \quad \delta^0 = \delta + \lambda \frac{1 - p_0}{1 - p}$$

- In the very first period, there are no preceding histories that can be affected by the change in shirking gains and thus $\delta = 0$.

We have shown that the shadow costs on the regeneration constraints, $\gamma$ and $\delta$ here, have an important role in the dynamic analysis. These capture the cost of changing their related variable on the preceding histories by aggregating the shadow costs of all the preceding ICs. In addition, as in every optimal contract, costs must equal benefits, these reflect the benefit of their related variable to the optimal continuation profit.

While the exposition used the LDIC problem, the only adjustment required for FDIC is to account for the additional ICs. This simply requires adding the related
terms. For example, the foc for $U^1$ in 2.6 is

$$p \cdot \frac{dV^1}{dU^1} + \sum_{s=0}^{n-1} \lambda_s \cdot (p - p_0) - p\gamma = 0.$$  

3.2. The Dynamic Dual Value

Suppose that before signing the contract, the principal and agent learn that some third party will have the power to stop the contract at history $h$ with some small probability $\varepsilon$. The dual problem is to determine the price per probability unit that the principal should be willing to pay to reduce $\varepsilon$ to zero. This price is the dual value of the history.

From the preceding analysis, the answer to this problem is clear – calculate the continuation profit less the costs imposed on the preceding histories. The dual value in a history $n$ with marginal utility and private gains costs of $\gamma$ and $\delta$ respectively has the following components:\footnote{It may be strange that the wage is not considered here directly. Wage is a way of providing utility. The dual value considers the commitment to provide the utility, rather than the method of actually providing it. The wage decision is captured in the next part of the problem.}

- The expected revenue increase: $v(p - p_0)$
- The marginal cost of the agent’s effort $c_n \cdot \lambda_0$ where $\lambda_0$ is the shadow cost of the LDIC. For simplicity again the exposition considers only LDICs.
- The utility cost of the agent’s effort on preceding histories $(-c_n) \cdot \gamma$ (payments and their effect on preceding histories are considered in the next section).
- The private gains cost of the agent’s effort on preceding histories, $d_n \cdot \delta$, reflecting that if the agent would have shirked in the past, his cost (resp. utility) today would have been lower (resp. higher) by $d_n$.
- The continuation dual values $p\mu^1 + (1 - p)\mu^0$, where superscripts indicate the current period’s outcome.

Formally, this third party termination is equivalent to an external reduction in the state variable $q$. The dual value will therefore be captured by $\mu$, the shadow price on the probability constraint (that is, $\frac{dV}{dq} = \mu$). As $q$ appears with no coefficient in the probability constraint, the dual value can be determined using the first order condition with respect to $q$. Applying recursion, and recalling we only consider the LDIC problem for which $s = 0$, obtains:
\[ q \text{ foc for LDIC: } \mu \geq v \cdot (p - p_0) - c_n \lambda_0 + c_n \gamma - d_n \delta_0 + p \mu^1 + (1 - p) \mu^0 \]

The next and final step verifies that the constraint will bind whenever \( \mu > 0 \) and determines the period shadow costs \( \lambda_s \) and the wage.

### 3.3. The Dual Problem

The dual value corresponds to the price the principal is willing to pay to prevent ex-ante, before proposing the contract, a forced termination in history \( h \). Generally, the dual solution of such maximization problems starts from the ‘best case’ outcome and reduces the value to satisfy all the constraints. The shadow cost of effort by an agent that shirked \( s \) times in the past is \( \lambda_s \) and in our setting, duality therefore implies that the foc 3.4 will bind and \( \lambda_s \) will be selected to minimize the RHS of the foc.

Focusing on the LDIC case with only one shadow cost, \( \lambda_0 \), the contract must find the ‘worst’ \( \lambda_0 \). If the agent would require no incentives for effort (\( \lambda_0 = 0 \)), the dual value would be highest and the principal would pay highly to prevent termination. Unfortunately, the agent requires any incentives he can get. The dual value should therefore the minimum of the right hand side of foc 3.4. In addition, as the principal can always simply agree to the termination, the dual value can never be negative.

Finally, the principal has one more tool at her disposal to curb the agent’s required incentives: payment for success. In particular, paying the agent relaxes the LDIC at a rate of \( \lambda (p - p_0) \) at a marginal cost to the principal of \( p \). In addition, paying increases the agent’s expected utility at a rate of \( p \), with an implied marginal cost to the principal \( p \cdot \gamma \). The optimal contract increases payment to the point that relaxing the LDIC is less profitable than the cost. That is,

\[ \text{LDIC wage constraint: } \lambda (p - p_0) \leq p + p\gamma \]

Problem 3.5 is therefore the FDIC dynamic dual (the LDIC dynamic dual is provided in section 4 after proving that LDIC are sufficient). The dynamic dual problem finds the shadow costs that minimize the dynamic dual value, subject to the marginal cost state variables derived above with their laws of motion and the wage constraint. The only difference from the LDIC dual is that all the FDICs and their shadow costs \( \lambda_s \) must be considered (again using \( \vec{\cdot} \) to denote vectors of length \( n - 1 \)). The non-
trivial effect of the FDIC is in the law of motion for the information rent. In addition to the rent from the additional cost difference that can be generated this period \( \frac{p_0}{p_0} \lambda_s \) as in the LDIC, the last term in the law of motion captures the rent in previous histories that was required to prevent the agent from shirking then and planning to shirk again now. The other changes from LDIC to FDIC are straightforward.\textsuperscript{14}

\[
\mu\left(n, \gamma, \delta \right) = \max_{s.t.} \left[ 0, \min_{\bar{\lambda} \geq 0} \mu\left(n, \gamma, \bar{\delta}, \bar{\lambda} \right) \right]
\]

\[
\mu\left(n, \gamma, \bar{\delta}, \bar{\lambda} \right) = v(p - p_0) + c_n \gamma - \sum_{s=0}^{n-1} (c_{n-s} \lambda_s + d_{n-s} \delta_s) + p \mu\left(n + 1, \gamma^1, \bar{\delta}^1 \right) + (1 - p) \mu\left(n + 1, \gamma^0, \bar{\delta}^0 \right)
\]

wage constraint: \( (p - p_0) \sum_{s=0}^{n-1} \lambda_s \leq p + p \gamma \)

utility costs: \( \gamma^1 = \gamma - \frac{p - p_0}{p} \sum_{s=0}^{n-1} \lambda_s \); and \( \gamma^0 = \gamma + \frac{p - p_0}{1 - p} \sum_{s=0}^{n-1} \lambda_s \)

information rent: \( \delta^1_s = \delta_s + \frac{p_0}{p} \lambda_s - \frac{p - p_0}{p} \sum_{j=s+1}^{n-1} \lambda_j \) and \( \delta^0_s = \delta_s + \frac{1 - p_0}{1 - p} \lambda_s + \frac{p - p_0}{1 - p} \sum_{j=s+1}^{n-1} \lambda_j \)

stopping condition: \( \mu\left(N^{FB} + 1, \gamma, \bar{\delta} \right) = 0 \)

**Proposition 1** Problem 3.5 is the dual of problem 2.6. In particular

\[
\mu\left(1, 0, 0 \right) = \max_{U,D \geq 0} V\left(1, 1, U, D \right) = \max_{U \geq 0} V\left(1, 1, U \right) \quad \text{and}
\]

the dual choice variables in every history are the FDIC shadow costs \( \lambda_s \) in both standard problems

\textsuperscript{14}In particular, for both the wage constraint and utility-cost law of motion replace \( \lambda_0 \) with the sum \( \sum_{s=0}^{n-1} \lambda_s \). For the objective, each \( \lambda_s \) multiplies the correct cost given the implies past number of shirks and the same for each \( \delta_s \).
ing it’s dual and then applying dynamic programming. If the primal dynamic problem is known to be concave, differentiable and feasible, the Slater condition (see e.g. Boro-
wein (2005)) can be used instead, following the steps outlined in the text above. The stopping condition reflects the fact that the principal can always simply agree to ter-
minate the contract. The more complicated notation is required only to accomodate the $s > 0$ private histories.

Because the state space reflects ‘costs’, this formulation has desirable properties, namely convexity and monotonicity. Intuitively, higher costs are always 'bad' and matter less as costs increase. In contrast, it is well known that in the standard formul-
ation higher utility for the agent may well be required to increase overall efficiency and the principal’s profits. Thus, the standard dynamic moral hazard problem is non-
monotonic in its state variable (the agent’s promised utility). These properties are exploited below to prove the one-shot-deviation (OSD) result, and later characterize the optimal contract.

**Proposition 2** The following hold for the dual problem 3.5:

- $\mu(n, \gamma, \delta)$ is convex in $(\gamma, \delta$).
- $\hat{\mu}(n, \gamma, \delta, \lambda)$ is convex in $\lambda$ for every $\gamma, \delta$.
- $\mu(n, \gamma, \delta)$ decreases in $\gamma$ and $\delta$ for any $s > 0$.

The first property is the mirror image of the concavity property of the standard dynamic moral hazard problem. The last property does not have a parallel property in the standard formulation. The continuation value for any period is almost never monotonic in the agent’s promised utility (the standard state variable). In contrast, the state variables $\gamma, \delta$ reflect the entire costs and the dual value $\mu$ reflects the full value of the sub-game starting at the history for the principal. As higher costs reduce the value, the second result is obtained.

4. **THE OPTIMAL CONTRACT**

4.1. *Sufficiency of LDIC*

This subsection establishes that it is sufficient to consider only local deviations (LDIC). The economic reason that only LDIC should bind in the optimal contract appears simple. If the agent did shirk in the past, his costs this period are lower ($c_n$ is increasing) and the effect of his shirking on future costs is lower ($d_n$ is increasing). Therefore, shirking in the past lowers the incentives to shirk and LDIC should be sufficient.
However, this intuition ignores a potential complication: if the agent expects to work more after failing than after succeeding, he may have a stronger incentive to shirk and fail so to enjoy the future lower costs may be larger. This is illustrated in the example used to prove lemma 2. There, the agent has an incentive to shirk in the first two periods, but not in any single period. Because the contract is such that the agent expects to work longer if he fails in both first periods, the private cost savings generated by the first shirk are more valuable if the agent increases the chances of failing in the second period, which is achieved by shirking in both periods.

Thus, to prove that LDIC are sufficient, we must show that the optimal contract, roughly, requires more work after success than after failure. However, this is not a typical result in dynamic moral hazard. For example, it is not clear whether the optimal contract requires more work after the sequence ‘fail, success, success’ or after the sequence ‘success, fail, success’ (see also the discussion at the end of section 2).

In contrast, the dual problem provides a direct proof that in the optimal contract, only the LDIC may bind. Intuitively, the dual problem can be thought of as considering, in each public history, which private history requires the strongest incentives. The solution determines the binding “Final Deviation Incentive Constraint” (FDIC) for each history.\footnote{This is similar to letting $s$ be an agent’s ‘type’ and identifying the ‘type’ that requires the strongest incentive to work.}

**Theorem 1** Any optimal contract subject to LDIC is an optimal contract.

The detailed proof is in appendix A.8. The following sketch identifies the economics underlying the result and the main technical steps.

The proof shows that if any FDIC binds ($\lambda_s > 0$), only the LDIC binds ($\lambda_{s>0} = 0$). This implies that only the LDIC matter for the optimal contract. In each period, the shadow prices on the FDIC $\lambda_s$, affect the dynamic dual value $\mu$ in the following ways:

- The period return must account for the agent’s cost ($-\lambda_sc_n-s$). As costs are highest if the agent never shirked, ($c_n > c_{n-s}$), the period return is minimized most by $\lambda_0$.

- The wage constraint accounts for the benefit of payment $\lambda_s (p - p_0)$. As the payment is worth the same for the agent regardless of his private history, the wage constraint is affected equally by all $\lambda_s$.\footnote{This is similar to letting $s$ be an agent’s ‘type’ and identifying the ‘type’ that requires the strongest incentive to work.}
The law of motion for next period’s incentive costs (γ^y and δ^y) are adjusted to account for the current period effects. These are analyzed in what follows.

Observing the law of motion in 3.5 confirms that the utility effect also does not depend on past shirking – all λs have the same impact on γ^y. This is because, as for the payment constraint, the agent values utility the same regardless of previous shirks.

However, any additional private gains depends greatly on the number of previous shirks. The law of motion for δ^s (the same applies to δ^0 with the obvious adjustments) shows this and that the shadow costs λs affects the information rent in two ways. First, the \( \frac{p_0}{p} \lambda_s \) term captures the rent from the additional cost difference that can be generated this period. Second, the \( \sum \frac{p - p_0}{1 - p} \lambda_j \) term captures the rent in previous histories that was required to prevent the agent from shirking then and planning to shirk now.

The first term captures the information rent cost ‘as if’ the agent could not control any previous decisions, while the second term adjusts the information rent cost to the fact that the agent did, in fact, made the previous decisions.

Because δ^s multiplies \( d_{n-s} \) in the period return, the first information rent effect \( \left( \frac{p_0}{p} \lambda_s \right) \) again has the most negative impact on the continuations by maximizing λ_0. This is the main role of convexity of costs (\( d_n \) increasing) in the proofs.¹⁶ For the second effect, convexity of the dual problem implies that reducing δ^1_s and increasing δ^0_s on equal terms increases the continuation value and therefore is not optimal. This last result is a direct benefit of using duality. The convexity of the dual problem implies that, in the optimal contract, costs after negative outcomes matter less. This is the formal analogue to the statement – ‘in the optimal contract, there is more work after success’, which was required to establish the sufficiency of local deviations.

### 4.2. The Optimal Contract Problem

Theorem 1 implies that the optimal contract can be derived considering only the LDIC. Both the standard problem (2.6) and the dual problem (3.5) can be simplified if only LDIC are considered. Both are useful to characterizing the optimal contract.

¹⁶The only other role is that convexity implies that the optimal contract is finite. It is possible however to assume sufficiently far-away finiteness independently (as in DeMarzo and Fishman (2007)). In addition, while applying duality in infinite horizon problems creates some challenges, there is room for research. See e.g. Anderson and Nash (1987) and Hernandez-Lerma and Hernandez-Hernandez (1994).
The standard (primal) LDIC problem is:

\[ (4.1) \]
\[
V(n, q, U, D) = \max_{(U^v, D^v, q, w) \geq 0} v(n, q, U^1, U^0, D^1, D^0) \\
\text{subject to} \\
v(n, q, w, U^1, U^0, D^1, D^0) = q(p - p_0) v - p \cdot w + pV(n + 1, q, U^1, D^1) + (1 - p)V(n + 1, q, U^0, D^0)
\]

Probability const. (μ) : \( q \leq \bar{q} \)

LDIC (λ) : \( w(p - p_0) - q \cdot c_n + (p - p_0)(U^1 - U^0) - p_0D^1 - (1 - p_0)D^0 \geq 0 \)

Regeneration -U (γ) : \( U = wp - qc_n + pU^1 + (1 - p)U^0 \)

Regeneration -D (δ) : \( D = qdn + pD^1 + (1 - p)D^0 \)

The dual LDIC problem is:

\[ (4.2) \]
\[
\mu(n, \gamma, \delta) = \max [0, \min_{\lambda \geq 0} \mu(n, \gamma, \delta, \lambda)] \\
s.t. \\
\mu(n, \gamma, \delta, \lambda) = v(p - p_0) - c_n\lambda + c_n\gamma - d_n\delta \\
+ p\mu(n + 1, \gamma^1, \delta^1) + (1 - p)\mu(n + 1, \gamma^0, \delta^0)
\]

wage constraint: \((p - p_0)\lambda \leq p(1 + \gamma)\)

utility cost: \(\gamma^1 = \gamma - \frac{p - p_0}{p}\lambda \; ; \; \gamma^0 = \gamma + \frac{p - p_0}{1 - p}\lambda\)

information rent: \(\delta^1 = \delta + \frac{p_0}{p}\lambda \; ; \; \delta^0 = \delta + \frac{1 - p_0}{1 - p}\lambda\)

stopping condition: \(\mu(N^{FB} + 1, \gamma, \delta) = 0\)

The following “standard” results allow a qualitative comparison between the standard and dual problems:

**Lemma 4** Problem 4.2 is convex in (γ, δ). The optimal solution decreases in the costs (γ and δ), γ, δ are substitutes. If \(\mu(n, \gamma, \delta, \lambda) < 0\), the contract is terminated. If \(\mu(n, \gamma, \delta, \lambda) = 0\), the optimal contract may randomize in the current period and then terminate regardless of outcome.

The dual problem has attractive features that the primal problem lacks. Because the dual state variables are “costs”, an increase in costs is always bad, and the problem
is monotonic. In addition, as one cost increases, the future becomes less attractive, and the other cost becomes less important. Thus, the dynamic dual problem has useful single-crossing properties that the standard problem lacks. Formally, $\mu(n, \gamma, \delta)$ is sub-modular in $(-\gamma, \delta)$\textsuperscript{17}.

As is well known, the standard continuation value may increase or decrease in the promised utility to the agent: if utility is too low, the agent can’t be asked to work, and if utility is too high, the agent must be provided all the remaining surplus.

In the standard model, the continuation value ($V$) does not account for the effect of the continuation on previous profits. As a result, it may very well have been optimal to commit to continue the contract despite a negative $V$, in order to increase profits in earlier periods. The dual value of a history, however, captures all the effects of the history on the ex-ante profit. Therefore, if it is negative, the contract terminates. Observing that the state variables $\gamma$ and $\delta$ are both higher in the continuation after failure than after success, obtains the following simple result:

**Lemma 5** If the contract terminates after success, it terminates after failure.

**Proof:** It is sufficient to show that $\mu$ after failure is no larger than $\mu$ after success. By lemma 4, this is the case if for every $\lambda$, $\gamma^0 \geq \gamma^1$ and $\delta^0 \geq \delta^1$. The first follows directly from $\lambda \geq 0$ : $\gamma^0 \geq \gamma \geq \gamma^1$. For the second, as $p > p_0$, we have that $\frac{1-p_0}{1-p} > 1 > \frac{p_0}{p}$ and so $\delta^0 \geq \delta^1$.

Note that the dynamic dual can also be derived for the “standard” problem in which costs are known (changing with $n$ or fixed). The dual procedure can then be applied and the result is the same problem without the $\delta$ variables. Thus lemma 4 (for $\gamma$ only) and lemma 5, apply also to the standard case.

### 4.3. Dynamic Quotas

This section provides the main characterization of the contract. We start with a common tradeoff in dynamic moral hazard problems – paying in wages vs. paying in continuation utilities. The dual analysis provides a simple proof for a general result. Note that the result also applies to the case that costs are known:

\textsuperscript{17}As the problem is convex, single crossing results rely on sub-modularity rather than super-modularity (which is used for the case that problems are concave).
Proposition 3  If the agent is ever paid in a period, the work plan \((q)\) and wage \((w)\) in all periods after the payment do not depend on future outcomes.

Proof: If the agent is paid in a period, \((w > 0)\), the dual wage constraint in that period must bind:

\[
\lambda = \frac{p + p\gamma}{p - p_0}
\]

Placing this in the continuation value for \(\gamma^1\) yields:

\[
\gamma^1 = \gamma - \frac{p - p_0}{p}\lambda = \gamma - \left( \frac{p + p\gamma}{p} \right) = -1
\]

Thus, in the next period after payment, the wage constraint is

\[
(p - p_0)\lambda \leq 0
\]

As \(\lambda \geq 0\), this constraint can be satisfied only by setting \(\lambda = 0\) which implies that the dual state variables in all continuation periods are fixed and cannot depend on additional outcomes, and so the contract cannot depend on these as well. Q.E.D.

Proposition 3 clarifies the tradeoff between incentivizing the agent through continuation utilities (conditioning the contract terms on future outcomes) and payments. The utility cost \(\gamma\), makes the distinction explicit. The proof of proposition 3 shows that if the contract decides to pay the agent in a period, it must be that any additional utility given to the agent after payment was used to generate some work and is essentially “free” to the principal. Formally, the preceding-histories utility cost in all future periods \((\gamma)\) after payment is \(-1\). The cost of paying the agent one util today is exactly compensated by the incentives this util generated for “free” in previous periods. Thus, the most profitable way to incentivize work from this point onwards is by simple payments.

The dual value \((\mu)\) for all remaining periods after payment is obtained by placing \(\gamma = -1\) and \(\lambda = 0\) in the dual objective:

\[
\mu(n, -1, \delta) = \max [v(p - p_0) - c_n - \delta d_n + \mu(n + 1, -1, \delta), 0]
\]

Proposition 4  If the agent is ever paid in a period, the contract in all periods after
the payment asks the agent to work until period $N(\delta)$ and pays the agent $\frac{c_{N(\delta)}}{p-p_0}$ for all remaining successes. $\delta$ is the information rent cost in the first period after payment and $N(\delta)$ is given by

$$N(\delta) = \max n : v(p - p_0) - c_n - \delta \cdot d_n \geq 0.$$  

The dual value to the principal is the remaining surplus less the information rent cost $\delta \cdot (c_{N(\delta)} - c_{n-1})$

**Proof:** The stopping period $N(\delta)$ is derived as the last period in which $\mu$ is still positive. To determine the wage, it is easiest to use the LDIC in problem 4.1. As the continuation utilities and work plans are fixed $U^1 = U^0$ and $D^1 = D^0 = q(c_{N(\delta)} - c_n)$.

The LDIC then simplifies to

$$w(p - p_0) - q \cdot c_n - q(c_{N(\delta)} - c_n) \geq 0$$

Or simply

$$\frac{w}{q} = \frac{c_{N(\delta)}}{p - p_0}.$$  

The agent’s wage for success in a period in which he works is exactly $\frac{w}{q}$. Q.E.D.

The dual value for a period is the efficient value $v(p - p_0) - c_n$ less the private information cost $\delta \cdot d_n$. The contract after payment does not depend on future outcomes, but does depend on previous outcomes, through $\delta$.

If costs are public information ($\delta = 0$), the optimal dynamic contract eventually either fires the agent without pay or “sells the firm” to the agent. Here, “selling the firm” means having the agent work in all remaining periods until the first best. It is easy to see that, if costs are public information, the contract can be implemented by paying the agent $\frac{c_n}{p - p_0}$ for all remaining periods until the first best period $N^{FB}$. At this last, first-best period $c_{N^{FB}} = v(p - p_0)$. The agent’s utility is exactly the principal’s fixed cost,\(^{18}\) and total surplus is zero. The dual value of the contract is exactly all the

\[^{18}\] $\frac{v(p - p_0)}{p - p_0} - v(p - p_0) = vp_0$
remaining surplus. That is, the dual value of the contract is highest after a payment was made.

While the dual value is maximal after payment, the standard continuation value may be negative. If $\delta$ is sufficiently low, $N(\delta)$ may even indicate the first best number of work periods and the principal’s expected payment to the agent may be larger than the expected revenue from work. This is not uncommon in dynamic moral hazard contracts – after “selling the firm to the agent” the principal’s value is negative. However, the dual value accounts for all the “free” work the agent provided to the principal in the preceding periods in anticipation of this reward and is in fact highest in this case. Ex-ante, the principal gains most from being able to eventually sell the firm to the agent than any other use of these continuation periods.

Proposition 4 describes the optimal contract as a “dynamic quota”. Once the agent meets a goal (his “quota”), he gets a fixed linear rate on all remaining sales. The quota here is dynamic because the goal and the expected linear rate potentially change until the agent does meet his quota. At the start of the contract and until he makes some quota, the agent may be rewarded only through the effect of outcomes on continuation utilities. Once the contract rewards the agent via payment, there is no going back – all future rewards are solely provided through payments.\(^{19}\)

The optimal contract therefore “ratchets” incentives. An agent that ‘proves’ his capabilities is paid more for working more. This is not the case when costs are known – once the agent gets paid, the continuation does not depend on the history at all. Thus, the combination of increasing and privately observed costs can provide a microeconomic foundation for ratcheting. Ratcheting has often been considered a common but misguided approach to incentives (see e.g. Weitzman (1980)). However, while suboptimal ratcheting may well be worse than a fixed contract, the contract identified here is an optimal ratcheting mechanism that outperforms any alternative contract.

If costs are public information, the optimal contract will, in some realizations be ex-post efficient. If the agent ever gets paid, he will work the first best number of periods. However, if costs are private the contract accumulates “private information costs” even if the agent succeeds in all periods. As a result, the contract must terminate before the first-best. The next proposition identifies the most efficient ex-post realization:

\(^{19}\)The distinction is stark because the agent is risk neutral. If the agent is risk averse, this would depend on the agent’s ability to save and borrow (see Fudenberg, Holmstrom, and Milgrom (1990)).
Proposition 5  In the optimal contract, the agent never works more for more than \( N \) periods.

\[
N = \max n : \quad v(p - p_0) \geq c_n + d_n \frac{p_0}{p - p_0}
\]

The agent works for exactly \( N \) periods if he never fails before the first payment. In particular, if \( v(p - p_0) < c_{NFB} + d_{NFB} \frac{p_0}{p - p_0} \) then the optimal contract is never ex-post efficient.

Another interesting result is that “second chances” are always worse for the agent. Suppose that the agent can make his quota by succeeding in history \( h \). That is, he would be paid for success in history \( h \), but was not paid in any history that precedes \( h \). If the agent fails, he may still meet his quota in a future history. However, the next result shows that the agent can never be ex-post better off from succeeding in such second chances, compared to succeeding at first. That is:

Lemma 6  If the agent is paid for a success in history \( h \), then any linear rate in any history that follows a failure in \( h \) is lower than the linear rate that starts in \( \langle h, 1 \rangle \).

In particular, if the agent is paid for success in history \( h \) and works in any history \( \langle h, 0, \tilde{h} \rangle \) then he works in the history \( \langle h, 1, \tilde{h} \rangle \).

Proof: By construction, \( \gamma = -1 \) in all histories of the form \( \langle h, 1, \tilde{h} \rangle \), and this is the lowest possible value for \( \gamma \). By the law of motion for \( \delta \),

\[
\delta^{(h,1,\tilde{h})} = \delta^{(h,1)} < \delta^{(h,0)} \leq \delta^{(h,0,\tilde{h})}
\]

As \( \mu \) decreases in both state variables, for any history \( \langle h, 0, \tilde{h} \rangle \), \( \mu^{(h,1,\tilde{h})} > \mu^{(h,0,\tilde{h})} \). Therefore, if the agent works \( \langle h, 0, \tilde{h} \rangle \) and would have been paid for success in history \( h \), he must also work in history \( \langle h, 1, \tilde{h} \rangle \). As the linear rate is set by the latest period of work, it cannot be larger after a failure than after success. \( Q.E.D. \)

Lemma 6 also establishes that if the agent would be paid following success, he cannot gain by ‘hiding’ the success from the principal and reporting it instead in a later period.\(^{20}\)

\(^{20}\)However, the lemma does not say anything about periods in which the agent would not be paid following success. Whether ‘hiding’ successes to be used in future reports can ever be profitable remains an open question here and, to the best of my knowledge, in the standard repeated moral hazard setting as well.
Another simple result that follows from proposition 3 is:

**Lemma 7** The IC always binds in the optimal contract.

**Proof:** In all periods in which the agent is paid, reducing the wage will increase profits and therefore the IC must bind. If the agent is not paid, it must be that the wage constraint does not bind. By complementary slackness the IC does not bind only if $\lambda = 0$. If the optimal contract sets $\lambda = 0$ the state variables in the next period are the same in all outcomes. Thus, the continuation contract and the agent’s expected continuation utility from the next period is the same regardless of outcomes. As $w_h$ is zero, the agent is not paid for success nor rewarded for his success in the future in any way. Therefore, the agent’s optimal plan must be to shirk in this period, contradicting incentive compatibility. \(Q.E.D.\)

5. EXTENSIONS, DISCUSSION AND IMPLEMENTATION

5.1. Higher Costs vs. Increasing Costs

A general presumption is that when it comes to costs, less is always better. However, lower costs in early periods generate a private information problem that may be more costly than the efficiency gain. A two period setting is sufficient to see this. If the optimal contract asks the agent to work in the second period only after success, the reduction in the difference between $c_1$ and $c_2$ increases the overall expected costs of effort. The increase in $c_1$ is paid in all possible realizations, but the second period cost decrease happens ex-post only with probability $p$. However, this also reduces the information problem. Proposition 6 shows that, under certain conditions, this increase in expected costs increases the contract’s expected profit.

**Proposition 6** In a two period problem, for any $\varepsilon \in (0, \frac{c_2 - c_1}{2})$ consider increasing $c_1$ by $\varepsilon$ and decreasing $c_2$ by $\varepsilon$. The change strictly increases expected profits if and only if:

- The optimal contract asks the agent to work after failing in the first period; or
- The optimal contract asks the agent to work only after success in the first period and $p + p_0 > 1$

The key to the proof is the following lemma, which provides a closed form solution to any two period problem:
Lemma 8  If the contract must terminate within at most two periods, the wage constraint binds in both remaining periods. In a two period problem, the dual values $\mu^0, \mu^1$ and $\mu^2$ are given by:

$$
\mu^1 = \max \left[ 0, v(p-p_0) - c_2 - \frac{p_0}{p-p_0}(c_2 - c_1) \right] 
$$

$$
\mu^0 = \max \left[ 0, v(p-p_0) + c_1 \frac{1-p_0}{p-p_0} \frac{p}{1-p} - c_2 \frac{p}{1-p} \frac{2-p}{p-p_0} \right] 
$$

$$
\mu = v(p-p_0) - \frac{p}{p-p_0} c_1 + p\mu^1 + (1-p) \mu^0 
$$

The only challenging part of lemma 8 is to determine that the wage constraint does indeed bind at all cases. Once that is obtained, the rest is simple algebra. Note that the first part of the lemma applies to any last two periods in a multi-period optimal contract.

If the optimal contract requires work in both periods, then by proposition 4, the agent’s wage is $\frac{c_2}{p-p_0}$ in both periods and $c_1$ does not matter for the principal’s expected profit. The closed form for $\mu^0$ shows that increasing $c_1$ and decreasing $c_2$ only increases $\mu^0$ and so the change proposed in the proposition will not change the optimal work plan but will decrease expected pay – increasing the expected profit.

If the optimal contract requires work in the first period and in the second period only after success

$$
\frac{\partial \mu}{\partial c_1} = -\frac{p}{p-p_0} + \frac{pp_0}{p-p_0} = -p(1-p_0) \quad \text{and} \quad \frac{\partial \mu}{\partial c_2} = -\frac{p^2}{p-p_0} 
$$

As in the previous case, the proposed change only increases $\mu^1$ and so the total work required by the optimal contract as a result of the change cannot decrease.

By the condition $p + p_0 > 1$, $\frac{\partial \mu}{\partial c_2} < \frac{\partial \mu}{\partial c_1}$, which proves proposition 6.

Even though second period utility is used to provide first period incentives and work in the second period will only be required ex-post with probability $p < 1$, the firm prefers a reduction in the second period cost to a reduction in the first period costs.

In contrast, if the costs are known (i.e. depend on calendar time), profits can only decrease from the change described in proposition 6.\textsuperscript{21}

\textsuperscript{21} An appendix with the dual closed form solution of the two period problem with known costs is available from the author.
5.2. Implementation Example

As the quote from Prendergast (1999) in the introduction suggests, current theory is challenged to explain frequently used mechanisms such as quotas and convex reward schemes. The following examples shows that the analysis so far can provide a micro-economic foundation for these. In particular, the example supports the use of “dynamic quota” contracts in dynamic settings.

Compared to regular quota or fixed rate contracts, the optimal contract increase profits by significantly reducing the agent’s expected utility. This is mostly done by giving up production after a series of bad outcomes, and increasing the pay rate after a series of good outcomes. The examples therefore support such ‘heuristic’ manipulations on regular quota contracts. In addition, these heuristic manipulations are consistent with the characteristics of convex reward schemes.

The examples also illustrate how the first outcomes, that require the least amount of effort, and do not generate any direct payment to the agent, have a bigger impact on the agent’s utility, as they generate higher information asymmetry.

The optimal contract was identified by implementing the dynamic program. The code is available from the author. Appendix C provides additional implementation details. Many other parametrizations were simulated. The examples described below illustrate the main qualitative conclusions. The optimal contract is compared to two commonly suggested contracts:

- The optimal quota contract that pays the agent $v$ for success starting from $m$-th success.
- The optimal linear contract that pays the agent $\frac{c_N}{p-p_0}$ for some $N$

These two contract types reflect the likely alternatives for dynamic settings – in which effort is exerted over time. Paying $v$ for success is consistent with the dynamic insight that utility should be provided as late as possible (see also Poblete and Spulber (2012)). The distinction between the two is somewhat semantic. The optimal linear contract can be implemented by a quota of zero and a payment of $\frac{c_N}{p-p_0}$ for success. Thus the linear contract is preferred to a quota contract only in extreme cases (see example 2).

The first example illustrates both the structure of the optimal contract and the effect of cost increases – the main topics of this paper. The example shows that the “dynamic quota” can be implemented as a quota contract with some relatively minor

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22I am not aware of explicit implementations of this theoretic mechanism.
adjustments in non-trivial settings.

A natural question to consider is the implication of having a dynamic setting. The second example below does this formally. Starting from a single period problem, production is systematically divided.

Both examples show significant returns for using the optimal dynamic contract relative to the two standard contracts. In contrast to profits, the optimal dynamic contract may not maximize surplus. In both examples, there are cases in which the quota or linear contracts generate a higher total surplus. This is expected as the inferior contracts increase the agent’s expected utility. Thus, from a social planner perspective, the returns from sophisticated contracts may not be as large.

5.2.1. First Example – Increasing the Cost Slope

The first example compares the effect of an increase in the cost slope for the three different contract types. Set $p = 0.6$, $p_0 = 0.2$, $v = 250$ so the expected revenue gain from work in a period is 100. There are at most nine periods. The average period cost is 50. Consider eleven possible linear cost curves so that the average is maintained but the slope varies from 0.1 to 10. That is, the first cost curve is $c_n = 49.6 + 0.1n$, the second cost curve is $c_n = 46 + n$, the third cost curve is $c_n = 42 + 2n$, etc. The cost curves are illustrated in figure 5.1. The first best surplus in all cases (obtained if the agent works in all periods) is 450.

Figure 5.1 presents the principal and the agent’s expected profit and the expected number of work periods from the optimal dynamic contract, the optimal quota contract and the optimal linear contract.

The main takeaway is that the optimal dynamic contract is between 35% to 50% more profitable than the most profitable quota contract, and as much as 250% more profitable than the linear contract. The quota contract here always outperforms the linear contract.

The difference between the optimal and quota contract is mainly in the agent’s expected utility. The quota contract rewards “lucky” agents (those that succeed without effort) more. In particular, in all quota contracts in this example, the agent shirks after

\[23\] A detailed comparison of the optimal quota and linear contract is outside the scope of the current discussion.
Figure 5.1. — Cost tilting example. The average effort cost in all cases is 50. In all cases $p = 0.6$, $p_0 = 0.2$ and $v = 250$. The top-left plot provides the effort cost per period as a function of $d_n$. The other plots present the expected profits, agent utility and work as a function of the cost difference.
two consecutive failures, waiting for a ‘lucky break’ before resuming work. In contrast, the optimal contract in all examples fires the agent after two failures. Allowing the agent to “stick around” and claim credit for the lucky successes increases his utility from failing. This requires stronger rewards for the earlier successes – effectively a transfer from the principal.

It is not surprising that quota contracts encourage slacking by unlucky agents. However, in the current example, slacking is generally expected. Overall, the agent is expected to shirk for 0.65 periods if costs are constant and this number increases to 1.37 periods if costs increase by 10 per period.

The optimal contract may not be very complicated. For example, if costs are $c_n = 46 + n$, the optimal contract is actually a quota of two with two additional clauses: The agent is fired if he fails in the first three periods or if he reaches the 8th period with only one success. These two small changes increase expected profits by about 50% compared to the best quota contract of three (from 230 to 350). In particular, the first change significantly reduces the agent’s expected utility from a start of two straight failures, which reduces the required compensation for success.

Figure 5.2 outlines the optimal contract for the case that $c_n = 10n$. The most prominent difference from the simple quota contract is that the quota threshold is determined after the first outcome is observed. In addition, the agent may be fired sooner than he’d like (given the quota) based on his outcomes. These relatively simple changes increase expected profits by 40% compared to the optimal quota contract.

A final interesting observation is that possibilities of cost-tilting may create organizational tensions. If the optimal mechanism is used, the principal prefers fixed costs while the agent prefers increasing costs. Thus the principal prefers a more balanced cost structure while the agent prefers to shift costs to later periods. However, if forced to use a linear contract, both the principal and the agent prefer fixed costs.

5.2.2. Second Example – The Dynamic Effect

To provide some intuition for the effect of dynamic settings, start from a setting in which cost is $c_n = 10n$, $p = 0.6$ $p_0 = 0.2$ and $v = 400$. The last efficient period is $N^{FB} = 16$ in which cost exactly equals the revenue increase from work. Now suppose that each adjacent period pair (i.e. 1-2, 3-4, etc.) is merged into one pair. That is, the two activities that formed periods one and two are now just one activity with a

\[24\text{This is qualitatively as complicated as the contracts get for this example.}\]
Figure 5.2.— Optimal contract for \(c_n = 10n\), \(p = 0.6\), \(p_0 = 0.2\), \(v = 250\). The agent’s expected continuation utility in the second period is 20 if he failed in the first period and 184 if he succeeded. The expected number of additional work periods starting in the second period is three if the agent failed in the first and 6.8 if the agent succeeded.

total cost of 30 (10 + 20). To keep surplus constant, each period’s success revenue is also doubled: \(v = 800\). This results in eight periods, with cost function \(c_n = 40n - 10\). Repeating the process results in a four-period problem with costs \(c_n = 160n - 60\) and \(v = 1600\), then a two period problem with \(c_n = 640n - 280\) and \(v = 3200\). One more final merge obtains a single period (i.e. static) problem with \(c_1 = 1360\) and \(v = 6400\).

In all of these configuration, the total expected revenue from efficient work is 0.4 \(\cdot 6400 = 2560\) and the total surplus is 1200 (= 2560 − 1360). In the single period setup, the optimal contract simply pays the agent \(\frac{1360}{0.4} = 3400\) for success and the principal’s expected profit is 2560 − 0.6 \(\cdot 3400 = 520\).

Figure 5.3 shows the optimal profits from the three considered contracts as a function of the number of periods the work is split over. As soon as there is some room for the optimal contract to condition on past performance, the optimal dynamic contract significantly out-performs the standard contracts. Even with work split over sixteen periods, the optimal contract is 20% more profitable than the quota contract. The low profits for the quota contract with a small number of period reflect that in those settings it is best to simply have a linear contract that is based on costs.

Interestingly, both the agent and the principal are better off by splitting the work as much as possible. Thus, in contrast to cost-tilting, which may generate organiza-
Figure 5.3.— Dynamic Effect example. The total effort cost in all cases is 1360. In all cases \( p = 0.6, p_0 = 0.2 \). Starting from 16 periods in which \( c_n = 10n \) and \( v = 400 \), the number of periods is cut in half, merging two adjacent periods into one while and doubling \( v \). The top-left plot provides the effort cost per period as a function of the total number of periods. The other plots present the expected profits, agent utility and work as a function of the cost difference, \( d_n \).
tional tensions, breaking down work to smaller tasks may be an easier organizational objective, provided that the contracting tools are relatively sophisticated (i.e. quota or optimal).

As in the previous example, the agent is expected to shirk in the quota contract following failures, and resume work after catching a “lucky break”. Still, this non incentive-compatible contract outperforms the linear contract if production takes place over a longer horizon.

6. CONCLUSION

This paper characterized the optimal contract in a dynamic moral hazard setting with persistent private information. The optimal contract problem was reformulated based on the two agency frictions – the cost to the principal of providing future utility ($\gamma$) and future private information ($\delta$) to the agent.

The resulting optimal contract was characterized as a dynamic quota. At the start of the contract the agent is not paid for successes. Once the agent is paid, he is paid a fixed linear piece-rate that depends only on his outcomes prior to the first payment.

The optimal contract explains features of real world contracts that puzzled economic observers. The variance in the expected total effort is larger with private cost information than without. Such large variation in ex-post incentives and effort across agents is inefficient and led several authors (see e.g. Oyer (1998); Larkin (2007); Misra and Nair (2009)) to suggest that there is significant room for improvement in either the design of real world incentives or models of the moral hazard setting. The model shows that this variance allows the firm to provide sufficient incentives for effort when it is relatively cheap and to provide high powered incentives when those are required without fear that agents misrepresent their effort (delay “easy sales” to the end of the period). The optimal contract must balance between efficiency (having the agents work longer) and profitability. While a high linear commission would guarantee all agents make the efficient level of effort, the firm’s profits would all be provided as rents. Consistent with the model, in the firm documented by Larkin (2007) the top end of the reward scale provides the salesperson a 25% commission on revenues, a figure very close to the industry’s accounting profit margins.

The analysis used arguments based on the duality of linear programs to design a dynamic program. The duality based analysis allows applying standard mechanism design techniques to the dynamic private information problem. The dynamic dual
problem has desirable features - namely monotonicity and single-crossing in the state space. These properties have so far been absent from dynamic moral hazard problems but are generally instrumental in the characterization of economic outcomes and comparative statics.

The use of duality in dynamic moral hazard problem has been advanced recently by Marcet and Marimon (2011) and Mele (2011) as well. Compared to these studies, the work here provides stronger and more direct results for a simpler and more specific framework. This allowed the dual formulation to provide new and important results that cannot be obtained using standard methods. In addition, the dual value of a period and the separation of the utility cost and the private information cost should prove useful for similar problems in future research.

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APPENDIX A: DETAILED PROOFS AND DERIVATIONS

A.1. Lemma 1

All contracts that are incentive compatible satisfy individual rationality. There is an optimal contract in which:

1. The agent works for at most \( N^{FB} \) periods
2. The required work decision is a stopping decision
3. The agent is never paid in a period with a failure or without required work

Proof: For generality, the proof will assume that there is a common discount factor \( \beta \in [0,1] \). All of the results are straightforward. The proof refers to a history \( h \) without defining what the history describes. It is sufficient to assume that \( h \) contains all the instructions given to the agent so far and all the outcomes observed.

Because the agent can choose to never work, limited liability implies that all contracts that are incentive compatible satisfy the participation constraint.

For 1: Take any contract in which the agent works for more than \( N^{FB} \) periods in some outcome realization. Choose one history \( \tilde{h} \) in which \( n > N^{FB} \) and the agent is asked to work. Let \( \tilde{u} \) be the agent’s expected continuation utility starting from \( \tilde{h} \). Because the agent can always not work in all remaining periods, \( \tilde{u} \geq 0 \). As surplus is negative, simply paying the agent \( \tilde{u} \) utils at the start of the history \( \tilde{h} \) and terminating the contract increases the expected profit.

For 2: Take any contract in which the agent is asked not to work in history \( h \) and expects to work in the next period with some probability \( \xi \in [0,1] \) (\( \xi \) may utilize the no-work period outcome as a public randomization device). An equivalent contract asks the agent to work in the current period with probability \( \xi \beta \) and terminates with probability \( 1 - \xi \beta \).

For 3:

- Take any history in which the agent is asked not to work but is paid. By the previous point, the contract terminates in this history and so this is not the first period. An equivalent contract pays the agent at the end of the previous period.
- Suppose the optimal contract pays the agent \( \xi > 0 \) for failing in the first period. A contract that does not pay \( \xi \) provides the agent stronger incentives to work. Thus, this new contract is also incentive compatible and provides the principal a higher expected profit. Because incentive compatibility implies individual rationality, the proof is complete.

\[25\text{This result may seem different from DeMarzo and Fishman (2006); Clementi and Hopenhayn (2006). In the setting there, the contract may continue for the maximal number of periods after randomization. However, the result there depends on the principal having a strictly positive outside option. If the outside option for the principal is worth zero, randomization there is never optimal.}\]
• Take any incentive compatible contract and any non-first history $\tilde{h}$ in which the agent is paid $\xi > 0$ after failure. Let $\tilde{q}$ be the probability that the agent is asked to work in the history $\tilde{h}$. Replace the contract with a contract that pays the agent $\xi (1 - p) \tilde{q}$ in the previous history if the agent obtains the outcome that leads to $\tilde{h}$. This relaxes the incentive constraint in history $\tilde{h}$ (as obtaining success is now more profitable than failing). There can be no effect on any other incentive constraint as the agent’s expected utility and work are not affected.

  - If the previous history required a success to lead into $\tilde{h}$, we are done.
  - If the previous history required a failure to lead into $\tilde{h}$, if this is the first history, we are done. If not, repeat the same process.

\[Q.E.D.\]

A.2. Lemma 2: LDIC does not imply IC

**Proof:** Suppose $c_n = n$, $p = \frac{1}{2}$ and $p_0 = 0$. We will show that the following contract violates IC but not LDIC:

• If the agent succeeds in any of the first two periods, he is paid 48 and is asked to stop working.
• If the agent fails in both first two periods, he is asked to work for eight more periods regardless of new outcomes and is paid 20 for each success.

First, we calculate $U^h_s$ for any period after the second. The agent is paid 20 for each remaining success and so:

$$U^h_s = \sum_{m=n_h}^{10} \frac{1}{2} \cdot 20 - \sum_{m=n_h}^{10} c_{m-s} = (11 - n_h) \cdot 10 - \sum_{m=n_h}^{10} c_{m-s}$$

In particular, if the agent failed in the first two periods, his expected utility from following the contract is:

(A.1) \[ U^{(0,0)}_s = 80 - 52 + 8s = 28 + 8s \]

Therefore, if the agent shirks in the first two periods, he is guaranteed to fail in those periods and his expected utility is $U^{(0,0)}_2 = 44$.

By complying, the agent’s expected utility is

$$U^0 = \frac{1}{2} \cdot 48 - 1 + \frac{1}{2} \left( \frac{1}{2} \cdot 48 + \frac{1}{2} U^{(0,0)}_0 - 2 \right) = \frac{1}{2} 48 - 1 + \frac{1}{2} 36 = 41$$

Therefore

$$U^0 < U^{(0,0)}_2$$

and the contract is not incentive compatible.
It remains to show that the contract does satisfy all LDIC: \textit{Q.E.D.}

- If the agent makes a first and last shirk in history $h$ with $n_h > 2$ he surely fails in that period and is expected utility is $U^{(h,0)}_1$. Thus, the LDIC for any $h$ with $n_h > 2$ is

$$U^{h}_0 \geq U^{(h,0)}_1.$$ 

Which is:

$$(11 - n) \cdot 10 - \sum_{m=n}^{10} c_m \geq (11 - (n + 1)) \cdot 10 - \sum_{m=n+1}^{10} c_{m-1}$$

Simplifying, the LDIC when $n_h > 2$ is

$$10 \geq \sum_{m=n}^{10} c_m - \sum_{m=n}^{9} c_m = c_{10}$$

As $c_n = n$, the LDIC binds in all periods starting from $n > 2$.

- Next consider period 2. To work in period 2, the agent must have failed in the first period. The LDIC is

$$\frac{1}{2} \cdot 48 + \frac{1}{2} \cdot U^{(0,0)}_0 - 2 \geq U^{(0,0)}_1$$

Using equation A.1, $U^{(0,0)}_0 = 28$ and $U^{(0,0)}_1 = 36$ so the LDIC strictly binds:

$$U^{0}_0 = 24 + 14 - 2 = 36$$

- Finally, for period 1, the LDIC is

$$\frac{1}{2} \cdot 48 + \frac{1}{2} \cdot U^{0}_0 - 1 \geq U^{0}_1$$

$U^{0}_0 = 36$ was obtained when the second period was considered. Therefore, the LHS of the LDIC is

$$U^{0} = \frac{1}{2} \cdot 48 + \frac{1}{2} \cdot 36 - 1 = 41$$

For $U^{0}_1$, if the agent shirked in the first period and complies starting from the second period then with probability $\frac{1}{2}$, a complying agent works only in period 2 and so gains just one more util from shirking before and with probability $\frac{1}{2}$, a complying agent works in all remaining periods and so gains nine more utils (one per period) from shirking before. Thus:

$$U^{0}_1 = U^{0}_0 + \frac{1}{2} + \frac{1}{2} \cdot 9 = 36 + 5 = 41$$

The LDIC in the first period binds as well.
Lemma 3: If a contract is FDIC it is IC

Proof: Suppose the contract $q, w$ is not IC. Then it violates FDIC:

1. As the set of possible work plans for the agent is finite and the agent’s expected profit is well defined and bounded for each work plan given $q, w$, there is a set $\hat{E}(q, w)$ of most profitable work plans given $q, w$.

2. Suppose $e^c \notin \hat{E}$ and let $\hat{e} \in \hat{E}$, be a most profitable deviating work plan.

3. Consider the set of histories $\hat{H}$ in which the agent makes a “final deviation” according to $\hat{e}$. That is, $\hat{h} \in \hat{H}$ if $q_{\hat{h}} > 0$, $\hat{e}_{\hat{h}} = 0$ and for every $h \succeq \hat{h}$, $h \neq \hat{h}$, either $q_h = 0$ or $\hat{e}_h = 1$. Let $\hat{s}$ be the number of past deviations at $\hat{s}$ according to $\hat{e}$. Clearly, if the agent profits from making this final deviation, the FDIC for $\hat{h}, \hat{s}$ is violated and the proof is complete.

4. If the agent does not profit from making this final deviation then the effort plan that complies in this last period provides at least the same expected profit to the agent. Thus, the effort plan with $\hat{e}_{\hat{h}} = 1$ provides at least the same expected profit for the agent. We can now repeat the process of searching for a profitable final deviation after setting $\hat{e}_{\hat{h}} = 1$. As $H$ is a finite set, the process ends either in finding a history in which FDIC is violated or if we change all periods in which $\hat{e}_h = 0$ to $\hat{e}_h = 1$ while weakly increasing the agent’s expected profit, implying that $\hat{e}$ was not more profitable than $e^c$.

$Q.E.D.$
A.4. Duality - Main Theorems

The classic reference is Dantzig (1963). The results are given in current textbooks on static optimization (see e.g. Vohra (2005)). Any linear problem may be written as

(A.2) \[ \max_{x \geq 0} c \cdot x \quad s.t. \quad Ax \leq b. \]

With \( c \) a vector of coefficients and \( A \) a matrix that holds in each row the coefficients on a constraint. The dual of the problem is

(A.3) \[ \min_{y \geq 0} y \cdot b \quad s.t. \quad yA \geq c. \]

The main results of interest are:

1. Each primal variable \( (x) \) translates to a constraint in the dual problem. Each primal constraint translates to a dual variable \( (y) \)
2. The Duality Theorem: If \( x^* \) and \( y^* \) are optimal, \( y^* \cdot b = c \cdot x^* \) whenever both exist and are finite; and
3. Complementary Slackness: \( y_i^* \) is the Lagrange multiplier in the primal solution for the constraint associated with the \( i \)-th row in \( A \). If \( y_i^* = 0 \) then the constraint associated with the \( i \)-th row in \( A \) does not bind when solving the primal.
4. The Dual of the Dual is the primal. Therefore, the primal variables \( x_i^* \) are the Lagrange multipliers in the dual’s solution.

The linearity of the objective implies:

1. If \( y_i^* = 0 \) then the solution to problem A.2 is not changed if the constraint associated with the \( i \)-th row in \( A \) is removed.

This last result is a combination of the Complementary Slackness result and the Fundamental Theorem of Linear Programming. See e.g. the discussion in Vohra (2005) preceding theorem 4.10 (Complementary Slackness).
A.5. Proposition 1 – Problem 3.5 is the dual of problem 2.6.

The proof first reconstructs problem 2.6 as a linear problem. Then derives the dual of the linear problem and finally reconstructs the dual as a recursive problem.

To save on notation, define \( P_h \) as the ex-ante probability that \( h \) is reached in equilibrium if the agent works in all periods up to \( h \).

**Lemma 9** Problem A.4 identifies the optimal contract:

\[
V^{FD} = \max_{q \geq 0, w \geq 0} \sum_{h \in H} P_h [q_h (p - p_0) v - w_h] \\
\text{s.t.} \quad q_h \leq 1, \quad \mu^0, \quad \mu^{(h,y)}, \quad \lambda^h
\]

With the FDIC:

\[
\text{FDIC} \ (h, s) : \quad -P_h (p - p_0) w_h + P_h q_h c_{h-s} \\
- (p - p_0) \left( \sum_{h \geq (h, 1)} \frac{P_h}{P} \left( p U_h - q_h c_h + q_h \cdot \sum_{i=0}^{s-1} d_{h-i} \right) \right) \\
+ (p - p_0) \left( \sum_{h \geq (h, 0)} \frac{P_h}{1-p} \left( p U_h - q_h c_h + q_h \cdot \sum_{i=0}^{s-1} d_{h-i} \right) \right) \\
+ p_0 \sum_{h \geq (h, 1)} \frac{P_h}{P} q_h \cdot d_{h-s} \\
+ (1 - p_0) \sum_{h \geq (h, 0)} \frac{P_h}{1-p} q_h \cdot d_{h-s} \leq 0
\]

**Proof:** The objective follows from 2.3. The probability constraint is the same as in problem 2.6, with both sides multiplied by \( P_h \) which is the strictly positive probability that history \( h \) occurs in equilibrium if . For the FDIC, start from the FDIC (2.5):

\[
p w_h - q_h c_h + p \left( U^{(h, 1)} + \sum_{j=0}^{s-1} D_j^{(h,1)} \right) + (1 - p) \left( U^{(h,0)} + \sum_{j=0}^{s-1} D_j^{(h,0)} \right) \\
\geq p_0 w_h + p_0 \left( U^{(h, 1)} + \sum_{j=0}^{s} D_j^{(h,1)} \right) + (1 - p_0) \left( U^{(h,0)} + \sum_{j=0}^{s} D_j^{(h,0)} \right)
\]

Collecting terms:

\[
(p - p_0) w_h - q_h c_h + (p - p_0) \left( U^{(h, 1)} + \sum_{j=0}^{s-1} D_j^{(h,1)} \right) \\
- (p - p_0) \left( U^{(h, 0)} + \sum_{j=0}^{s-1} D_j^{(h,0)} \right) \\
-p_0 D_s^{(h, 1)} - (1 - p_0) D_s^{(h, 0)} \geq 0
\]

Recall that \( U^h \) is the agent’s continuation utility starting at the public history if the agent complied in all histories and continues to comply. That is:

\[
U^h = p w_h - q_h c_h \cdot p U^{(h, 1)} + (1 - p) U^{(h, 0)}
\]

\[
\sum_{h \geq h} \frac{P_h}{P} \left[ p w_h - q_h c_h \right]
\]

(A.6)
Similarly, \( D^h \) is the expected continuation cost saving

\[
D^h_s \equiv q_h d_{h-s} + p D^{(h,1)}_s + (1 - p) D^{(h,0)}_s
\]

(A.7)

\[
= \sum_{h \geq h} \frac{p^h}{p^h} q_h d_{h-s}
\]

Finally

\[
(A.8) \sum_{j=0}^{s-1} D^h_j = \sum_{h \geq h} \frac{p^h}{p^h} q_h \left( \sum_{j=0}^{s} d_{h-j} \right)
\]

Placing equations A.6, A.7 and A.8 in the FDIC and multiplying by \( P_h \) obtains the linear FDIC A.5.

\[Q.E.D.\]

**Lemma 10** Problem A.9 is the dual of problem A.4.

(A.9) \[
\min_{(\mu, \lambda) \geq 0} \mu^0 \\
\text{s.t. } \forall h: \\
\text{Wage (} w_h \text{) constraint: } (p - p_0) \sum_{s=0}^{n_h-1} \lambda^h_s \leq p + p \cdot \sum_{s=0}^{n_h-1} \gamma^h_s
\]

\[
q_h \text{ constraint: } \mu^h \geq v (p - p_0) + (1 - p) \mu^{(h,0)} + p \mu^{(h,1)} \\
+ \gamma^h \cdot c_h - \sum_{s=0}^{n_h-1} (\lambda^h_s \cdot c_{h-s} + \delta^h_s \cdot d_{h-s})
\]

\[\text{stopping condition: } n_h > N^{FB} \implies \mu^h = 0\]

and

\[
\gamma^h \equiv - (p - p_0) \left[ \sum_{h \geq h} \frac{\lambda^h_s}{p} - \sum_{h \geq h} \frac{\lambda^h_s}{1 - p} \right]
\]

\[
\delta^h_s \equiv \sum_{h \geq h} \left( \frac{p_0}{p} \lambda^h_s - \frac{p_0}{p} \sum_{j=s+1}^{n_h-1} \lambda^h_j \right) + \sum_{h \geq h} \left( \frac{1 - p_0}{1 - p} \lambda^h_s + \frac{p - p_0}{1 - p} \sum_{j=s+1}^{n_h-1} \lambda^h_j \right)
\]

**Proof:** The objective \( \mu^0 \) is straightforward as it is the multiplier on the only constraint with non-zero RHS. The stopping condition follows as \( q_h = 0 \) is the optimal solution whenever \( n_h > N^{FB} \) and so the probability constraint cannot bind.

It remains to derive the constraints for each \( w_h \) and \( q_h \).

- For \( w_h \), we show that the constraint is

\[
-(p - p_0) \sum_{s=0}^{n_h-1} \lambda^h_s + p \cdot \gamma^h \geq -p
\]
The right hand side of the constraint is the coefficient on \( w_h \) in the objective of \( V^{FD} \):

\[-P_h p.\]

\( w_h \) appears with a coefficient \(-P_h (p - p_0)\) in all the FDIC (A.5) for \( h \). Each has a shadow variable \( \lambda^h_s \). Summing obtains the first element of the left hand side

\[-P_h (p - p_0) \sum_{s=0}^{n_h-1} \lambda^h_s\]

The variable \( w_h \) also appears with a coefficient \(-P_h p (p - p_0)\) in all FDIC (A.5) for \( \hat{h} \) such that \( h \succeq \langle \hat{h}, 1 \rangle \) and with a coefficient \( P_h \frac{p}{1-p} (p - p_0) \) in all FDIC for \( \hat{h} \) such that \( h \succeq \langle \hat{h}, 0 \rangle \).\(^{26}\) Taking \( P_h p \) out and summing up, this yields:

\[P_h p \cdot \sum_{s=0}^{n_h-1} (p - p_0) \left[ -\sum_{h:h \geq \langle \hat{h}, 1 \rangle} \frac{\lambda^h_s}{p} + \sum_{h:h \geq \langle \hat{h}, 0 \rangle} \frac{\lambda^h_s}{1-p} \right] = P_h p \cdot \gamma^h\]

Dividing all three previous bullet points by \( P_h \) obtains the result.

- For \( q_h \), we show that the constraint is

\[P_h \cdot \left[ \mu^h - (1 - p) \mu^{(h,0)} - p \mu^{(h,1)} + \gamma^h c_h - \sum_{s=0}^{n_h-1} \left( \lambda^h_s \cdot c_{h-s} + \delta^h_s \cdot d_{h-s} \right) \right] \geq P_h v (p - p_0)\]

Then, dividing both sides by \( P_h \) and isolating \( \mu^h \) obtains the result.

- The variable \( q_h \) appears in the objective with coefficient \( P_h v (p - p_0) \), obtaining the right hand side.

- \( q_h \) appears in the three probability constraints. These generate the first three terms in the left hand side

\[(A.10) \quad P_h \mu^h - P_h (1 - p) \mu^{(h,0)} - P_h p \mu^{(h,1)}\]

- In all the FDIC for history \( h \) (i.e., for each \( s \)), \( q_h \) appears with a coefficient \( c_{h-s} \). This generates the term

\[\sum_{s=0}^{n_h} P_h c_{h-s} \lambda^h_s.\]

- The variable \( q_h \) also appears three times in each of the FDIC for \( \hat{h}, s \) such that \( h \succeq \hat{h} \). Once as part of the continuation utility term multiplying \( c_h \) (in either the second or third row of A.5) and twice as part of the future gains from shirking term. The continuation

\(^{26}\)Note that the history \( h \) here is \( \hat{h} \) in those relevant FDIC (A.5) and the history \( h \) in the FDICs is the history \( \hat{h} \) here.
utility term will determine the coefficient on \( \gamma \). The shirking gains term will determine the coefficients on \( \delta \).

1. In the continuation utility term in the FDIC for all histories that \( h \) follows, the coefficient for \( q_h \) in the FDIC for \( (\bar{h}, s) \) is multiplied by the public cost this period \((c_h)\) and \( P_h \frac{p-p_0}{1-p} \) if \( h \succeq (\bar{h}, 1) \) or \( -P_h \frac{p-p_0}{1-p} \) if \( h \succeq (\bar{h}, 0) \). Summarizing these terms obtains the sum:

\[
A.11 \quad P_h \sum_{s=0}^{n_h} c_{h} (p - p_0) \left[ \sum_{\bar{h} : h \succeq (\bar{h}, 1)} \frac{\lambda^h}{p} - \sum_{\bar{h} : h \succeq (h, 0)} \frac{\lambda^h}{1-p} \right] .
\]

Simple algebra yields the term

\[ -P_h \gamma^h c_h \]

2. In the continuation utility term in the FDIC for all histories that \( h \) follows, the coefficient for \( q_h \) in the FDIC for \( (\bar{h}, s) \) is multiplied by the private cost difference in history \( (h, s) \): \( \left( \sum_{j=0}^{s-1} d_{h-j} \right) \) and \( -P_h \frac{p-p_0}{1-p} \) if \( h \succeq (\bar{h}, 1) \) or \( P_h \frac{p-p_0}{1-p} \) if \( h \succeq (\bar{h}, 0) \). Adding:

\[
A.12 \quad P_h (p - p_0) \sum_{s=1}^{n_h-1} \left[ \sum_{\bar{h} : h \succeq (\bar{h}, 1)} \frac{\lambda^h}{p} \left( \sum_{j=0}^{s-1} d_{h-j} \right) - \sum_{\bar{h} : h \succeq (h, 0)} \frac{\lambda^h}{1-p} \left( \sum_{j=0}^{s-1} d_{h-j} \right) \right]
\]

3. In the information rents terms in each FDIC constraint (the last two rows in A.5), the coefficient for \( q_h \) is \( P_h \frac{p_0}{p} d_{h-s} \) if \( h \succeq (\bar{h}, 1) \) and \( \frac{1-p_0}{1-p} P_h d_{h-s} \) if \( h \succeq (\bar{h}, 0) \). Again the coefficients depend only on whether \( h \) follows a success or failure in \( \bar{h} \). Adding up all the relevant terms obtains:

\[
A.13 \quad P_h \sum_{s=0}^{n_h-1} d_{h-s} \left[ \frac{p_0}{p} \sum_{\bar{h} : h \succeq (\bar{h}, 1)} \lambda^h + \frac{1-p_0}{1-p} \sum_{\bar{h} : h \succeq (h, 0)} \lambda^h \right] .
\]

Adding the last two terms A.12 and A.13 obtains

\[
P_h \sum_{s=0}^{n_h-1} \delta^h d_{h-s} \]

Q.E.D.

Lemma 11  In the optimal solution to the dual problem A.9

\[
\mu^h = \max \left[ 0, v (p - p_0) + (1 - p) \mu^{(h,0)} + p \mu^{(h,1)} + \gamma^h c_h - \sum_{s=0}^{n_h-1} (c_{h-s} \lambda^h + d_{h-s} \delta^h) \right]
\]
Proof: By construction, $\mu^h \geq 0$. The lemma states that if $\mu^h > 0$ then

$$\mu^h = v(p - p_0) + (1 - p)\mu^{(h,0)} + ph^{(h,1)} + \gamma^h c_h - \sum_{s=0}^{n_h-1} (c_{h,s}\lambda^h_s + d_{h,s}\delta^h_s)$$

Suppose the statement is false.

1. If $\mu^\emptyset$ violates the condition, decrease $\mu^\emptyset$. This is feasible and decreases the objective. Therefore, $\mu^\emptyset$ was not optimal.

2. If any other history violates the condition, there must be an $h$ that violates the condition and that in all histories that precede $h$ the condition holds. Decrease $\mu^h$ by $\varepsilon$. As the constraint in the previous period binds, this allows decreasing the previous history’s $\mu$ by either $\varepsilon \cdot p$ or $\varepsilon (1 - p)$. Continuing backwards, this will decrease $\mu^\emptyset$. This is a feasible decreases of the objective.

Q.E.D.

Lemma For every $h$ let $\mu^h, \lambda^h$ be the solution to the dual problem A.9. Then $\mu^h = \mu(n_h, \gamma^h, \delta^h)$ and $\lambda^h \in \lambda^*(n_h, \gamma^h, \delta^h)$. In particular, $\mu(1,0,0) = \mu^\emptyset$. Where $\mu(n, \gamma, \delta)$ is defined recursively in 3.5.

Proof: First, observe that the definitions in A.9 for $\gamma^h$ and $\delta^h$ yield the recursive definition in 3.5.

Next, for any history $h$, let $\sigma^h$ be the solution to the problem of minimizing $\mu^h$ subject to the constraints in problem A.9 for all histories that follow $h$ and given the $\delta^h$ and $\gamma^h$ that correspond to the problem’s solution starting at $\mu^\emptyset$. The solution $\sigma^h$ must be identical on all common histories to the solution for the original problem. In addition, the solution $\sigma^h$ only specifies variables for histories that follow $h$.

Applying the Principle of Optimality, any problem starting at history $h$ can be broken down to choosing the $\lambda^h_s$ in history $h$ and optimizing the continuation problem. It can be verified that the constraints for the recursive formulation are identical to the constraints in the linear formulation. Lemma 11 proves that the value of the solution in all continuations is the one defined in the objective. Q.E.D.

A.6. Proposition 2 – Basic Properties of the Dual

Proposition The following hold for the dual problem 3.5:

- $\mu(n, \gamma, \delta)$ is continuous and convex in $(\gamma, \delta)$. $\mu(n, \gamma, \delta, \lambda)$ is continuous and convex in $\lambda$ for every $\gamma, \delta$.
- $\mu(n, \gamma, \delta)$ decreases in $\gamma$ and $\delta_s$ for any $s \geq 0$.

Proof: We start with convexity:

- In any last period, $\mu(n, \gamma, \delta)$ is linear in $\gamma, \delta$ and thus continuous and convex.
• Assume that $\mu(n+1, \gamma, \delta)$ is continuous and convex. The positive sum of three continuous and convex functions is continuous and convex. Thus, for every $\lambda$, $\mu(n, \gamma, \delta, \lambda)$ is continuous and convex in $(\gamma, \delta)$. As the feasible set is convex and the objective is to minimize a convex function, $\mu(n, \gamma, \delta)$ is continuous and convex.

• For $\mu(n, \gamma, \delta, \lambda)$, the period return is linear in all $\lambda$ and so it is sufficient to show that the continuation is convex in $\lambda$. As the continuations are all convex in $\gamma$ and $\delta$, this follows from $\gamma^y$ and $\delta^y$ all being linear functions of $\lambda$.

For decreasing: For simplicity of notation, the proof uses subscripts of functions to denote the partial derivative if it exists.

Q.E.D.

1. $\mu(n, \gamma, \delta)$ decreases in $\delta_s$ for any $s \geq 0$. By backward induction:

• In any last period if $\mu(n, \gamma, \delta) > 0$ then $\mu_{\delta_s} = -d_{n-s} < 0$. If $\mu(n, \gamma, \delta) < 0$ then $\mu_{\delta_s} = 0$.

At $\mu = 0$ an decrease in $\delta_s$ increases $\mu$ by $d_{n-s} > 0$

• For any non-last period, it must be that $\mu(n, \gamma, \delta) > 0$. Suppose $\mu(n+1, \gamma, \delta)$ decreases in $\delta_s$. Let $\lambda^*$ be optimal at the state $(n, \gamma, \delta)$ and let $e_s$ be a vector of size $n-1$ with the $s$-th element some $\varepsilon > 0$ and all other elements zero. As the constraint is not affected by $\delta$, $\lambda^*$ is feasible for any $(n, \gamma, \delta + e_s)$. The additional $e_s$ decreases the current period return by $d_{n-s}$ and increases the continuation $\delta_s$. Therefore, it decreases $\mu(n, \gamma, \delta)$.

2. $\mu(n, \gamma, \delta)$ decreases in $\gamma$

• In any last period, $\mu$ decreases most in $\lambda_0$ and thus the constraint must bind and

$$\begin{align*}
\lambda_0 &= \frac{p}{p - p_0} (1 + \gamma) \\
\lambda_{s>0} &= 0
\end{align*}$$

Thus, in any last period

$$\mu(n, \gamma, \delta) = v(p-p_0) + c_n\gamma - c_n\frac{p}{p - p_0} (1 + \gamma) - \sum_{s=0}^{n-1} \delta_s d_{n-s}$$

If $\mu > 0 \mu_\gamma = -p_0 \frac{c_n}{p - p_0} < 0$. If $\mu < 0 \mu_\gamma = 0$ and if $\mu = 0$ these two values bound the super-gradient.

• For any non-last period with state $(n, \gamma, \delta)$, let $\lambda^*$ be an optimal solution. Let $\gamma^y$ and $\delta^y$ be the corresponding continuation states. An increase $\varepsilon$ in $\gamma$ makes it feasible to increase $\lambda_0$ by $\varepsilon \frac{p}{p - p_0}$. Denote $\tilde{\gamma}^y$ the new continuation $\gamma$:

$$\begin{align*}
\tilde{\gamma}^1 &= \gamma^1 + \varepsilon - \frac{p - p_0}{p} \varepsilon \frac{p}{p - p_0} = \gamma^1 \\
\tilde{\gamma}^0 &= \gamma^0 + \varepsilon + \frac{p - p_0}{1 - p} \varepsilon \frac{p}{p - p_0} = \gamma^0 + \frac{\varepsilon}{1 - p} > \gamma^0
\end{align*}$$

In addition, both $\delta^1_0$ and $\delta^0_0$ increase while the remaining $\delta^y_s$ are unchanged. From the previous part, $\mu(n+1, \gamma, \delta)$ decreases in all $\delta_s$. By the induction assumption $\mu(n+1, \gamma, \delta)$
also decreases in $\gamma$ and so

$$\mu \left( n + 1, \gamma^y, \delta^y \right) \leq \mu \left( n + 1, \gamma^y, \delta^y \right)$$

Therefore:

$$\mu \left( n, \gamma + \varepsilon, \delta \right) \leq p\mu \left( n + 1, \gamma^1, \delta^1 \right) + (1 - p) \mu \left( n + 1, \gamma^0, \delta^0 \right)$$

$$v \left( p - p_0 \right) + c_n \left( \gamma + \varepsilon \right) - c_n \varepsilon \cdot \frac{p}{p - p_0} - \sum_{s=0}^{n-1} \left( c_{n-s} \lambda_s^* + d_{n-s} \delta_s \right)$$

$$\leq \mu \left( n, \gamma, \delta \right) + c_n \varepsilon \left( 1 - \frac{p}{p - p_0} \right)$$

$$< \mu \left( n, \gamma, \delta \right)$$

A.7. Lemma 4

**Lemma** Problem 4.2 is convex in $(\gamma, \delta)$. The optimal solution decreases in the costs $(\gamma$ and $\delta)$, $\gamma, \delta$ are substitutes. If $\mu \left( n, \gamma, \delta, \lambda \right) < 0$, the contract is terminated. If $\mu \left( n, \gamma, \delta, \lambda \right) = 0$, the optimal contract may randomize in the current period and then terminate regardless of outcome.

**Proof:** The proof for the first two statements (convex and decreasing) is identical to the proof of proposition 2. If $\mu \left( n, \gamma, \delta, \lambda \right) < 0$ then in the optimal contract problem the constraint $q \geq 0$ for the history binds. Thus, $q = 0$ and the contract is terminated. If $\mu \left( n, \gamma, \delta, \lambda \right) = 0$ neither constraint for $q$ bind, implying some randomization, except for pathological cases in which the constraint “weakly binds”. From Lemma 1, the contract terminates after the randomization.

As $\mu$ is decreasing in $\gamma, \delta$, they are substitutes if the first difference (i.e. derivative or sub-gradient) of $\mu$ w.r.t. $\gamma$ is increasing in $\delta$ (and vice versa). Without assuming differentiability, this is equivalent to requiring that $\mu$ is sub-modular in $(-\gamma, \delta)$. We use $\mu_\gamma$ to denote the first difference (i.e. derivative or sub-gradient). We show the condition holds by backward induction:

1. In any last period $n$, the derivative of $\mu_\gamma = -p_0 \frac{c_n}{p - p_0} < 0$ if $\mu > 0$ and zero otherwise (see proof for proposition 2). $\mu$ decreases in $\delta$. Thus, when $\delta$ increases sufficiently, $\mu_\gamma$ increases to zero. The same argument applies for $\mu_\delta$.

2. In any non-last period, the feasible set is a sub-lattice (only $\gamma$ and $\lambda$ are in the constraint).
   - The space $X \left( y \right) \times Y$ is a sub-lattice iff for any $x \in X \left( y \right)$ and $x' \in X \left( y' \right)$, $x \land x' \in X \left( y \land y' \right)$ and $x \lor x' \in X \left( y \lor y' \right)$

Q.E.D.

A.8. Sufficiency of Local Deviations

**Theorem** Any optimal contract subject to LDIC is an optimal contract.

By complementary slackness it is sufficient to show that if for any $s$, $\lambda_s > 0$, then it must be that only $\lambda_0 > 0$. Suppose that for some $j > 0$, $\lambda_j = \varepsilon > 0$. Consider setting $\tilde{\lambda}_j = 0$ for all $j > 0$ and
\[ \lambda_0 = \sum_{s=0}^{n} \lambda_s. \] That is, the period return is lower as \( c_n > c_{n-s} \) for any \( s > 0 \). The wage constraint and utility cost are unaffected as they both depend on the sum of the period shadow costs \( \sum_{s=0}^{n-1} \lambda_s \).

Thus, it remains to consider only the effect on the information rent, through the law of motion for \( \delta \). As the shadow costs \( \lambda \) affect \( \delta \) in two ways, the analysis is simplified by splitting \( \delta \) into two variables, \( \xi \) and \( \chi \) such that \( \delta = \xi + \chi \):

\[
\xi_s^1 = \xi_s + \frac{p_0}{p} \lambda_s; \quad \xi_s^0 = \xi_s + \frac{1-p_0}{1-p} \lambda_s
\]
\[
\chi_s^1 = \chi_s - \frac{p-p_0}{p} \sum_{j=s+1}^{n-1} \lambda_j; \quad \chi_s^0 = \chi_s + \frac{p-p_0}{1-p} \sum_{j=s+1}^{n-1} \lambda_j
\]

We show that in both cases decreasing all \( \lambda_s \) to zero and increasing \( \lambda_0 \) by \( \sum_{s=1}^{n} \lambda_s \) reduces the continuation values. Applying the proof of proposition 2, it is immediate that \( \mu(n, \gamma, \xi, \chi) \) is convex and decreasing in \((\gamma, \xi, \chi)\). Now:

- For any \( s > 0 \), decreasing \( \lambda_s \) and increasing \( \lambda_0 \) by the same amount is equivalent to decreasing \( \xi_s^0 \) by \( \epsilon \cdot \xi_s^0 \) and increasing \( \xi_s^0 \) by \( \epsilon \cdot \xi_s^0 \). It is sufficient to show that if \( \xi_s \geq \epsilon > 0 \) then \( \mu(n, \gamma, \xi, \chi) \) decreases if \( \xi_0 \) is replaced by \( \tilde{\xi}_0 = \xi_0 + \epsilon \) and \( \xi_s \) is replaced by \( \tilde{\xi}_s = \xi_s - \epsilon \). The constraint on \( \lambda \) in any period is unaffected by the change so it is sufficient to show that for any optimal \( \lambda \), the change decreases \( \mu \)
  - In any last period, the change decreases \( \mu \) by \( \epsilon \cdot (d_n - d_{n-s}) > 0 \). The sign follows from \( d_n \) increasing.
  - In any period \( n \) that is not last, suppose that the continuation values decrease from such a change. The current period return is decreased by \( \epsilon \cdot (d_n - d_{n-s}) > 0 \) and all the continuation values are decreased.

- For \( \chi_s \), the change would set all \( \chi_s \) to zero. In particular, let \( k_s = (p-p_0) \sum_{j=s+1}^{n-1} \lambda_j \) and it is sufficient to show that
  \[
p \cdot \mu \left( n, \gamma^1, \xi^1, \chi^1 - \frac{k_s}{p} \right) + (1-p) \mu \left( n, \gamma^0, \xi^0, \chi^0 + \frac{k_s}{1-p} \right)
\]
  is decreasing in any \( k_s \). This follows from \( \gamma^1 < \gamma^0, \xi^1 < \xi^0, k_s > 0 \) and \( \mu(n, \gamma, \xi, \chi) \) convex and decreasing in \((\gamma, \xi, \chi)\).

A.9. Proposition 5

**Proposition** In the optimal contract, the agent never works more for more than \( \bar{N} \) periods.

\[
\bar{N} = \max n : \quad v(p-p_0) \geq c_n + d_n \frac{p_0}{p-p_0}
\]

The agent works for exactly \( \bar{N} \) periods if he never fails before the first payment. In particular, if \( v(p-p_0) < c_{N^{FB}} + d_{N^{FB}} \frac{p_0}{p-p_0} \) then the optimal contract is never ex-post efficient.

**Proof:** The linear contract with the highest \( N(\delta) \) is the linear contract with the lowest possible information rent \( \delta \). As \( \frac{p_0}{p} < \frac{1-p_0}{1-p} \), the lowest \( \delta \) for any specific sequence of \( \lambda \)'s is obtained if the
agent constantly succeeds:

\[ \delta^h \geq \frac{p_0}{p} \sum_{h \geq h} \lambda^h \]

The inequality is an equality if the agent never failed in the past.

At the start of the contract \( \gamma = 0 \). For the contract to move to the linear rate it must be that \( \gamma^h = -1 \). By the law of motion for \( \gamma \), this implies that the sequence of past \( \lambda \) must satisfy at least

\[ \frac{p - p_0}{p} \sum_{h \geq h} \lambda^h \geq 1 \cdot \]

The inequality is an equality if the agent never failed in the past. In this case

\[ \sum_{h \geq h} \lambda^h = \frac{p}{p - p_0} , \]

and the lowest possible value for \( \delta^h \) is:

\[ \delta^h = \frac{p_0}{p} \frac{p}{p - p_0} = \frac{p_0}{p - p_0} . \]

Placing this in \( N(\delta) \) yields the desired result. \( Q.E.D. \)

A.10. Two Period Solution

First prove the lemma:

**Lemma** If the contract must terminate within at most two periods, the wage constraint binds in both remaining periods. In a two period problem, the dual values \( \mu^1, \mu^0 \) and \( \mu^1 \) are given by:

\[
\mu^1 = \max \left[ 0, v(p - p_0) - c_2 - \frac{p_0}{p - p_0} (c_2 - c_1) \right] \\
\mu^0 = \max \left[ 0, v(p - p_0) + c_1 - \frac{p_0}{p - p_0} (1 + c_2) - \frac{2 - p}{1 - p} \right] \\
\mu = v(p - p_0) - \frac{p}{p - p_0} c_1 + p \mu^1 + (1 - p) \mu^0
\]

**Proof:** In any last period, the dual problem is:

\[
\mu(n, \gamma, \delta, \lambda) = \min_{\lambda \geq 0} v(p - p_0) - c_n \lambda - \delta d_n + \gamma \cdot c_n \\
s.t. \quad (p - p_0) \lambda \leq p(1 + \gamma)
\]

The objective decreases with \( \lambda \) (by \(-c_n\)) and therefore must bind in the optimal solution. Placing \( \lambda \) in the objective, simplifying and binding by zero

\[
\mu(n, \gamma, \delta) = \max \left[ 0, v(p - p_0) - \frac{p}{p - p_0} - \delta d_n - \gamma \cdot c_n \frac{p_0}{p - p_0} \right]
\]
The first two terms are the ‘static return’ the next term is the private information cost, and the last term is the utility cost \((c_n \cdot \frac{p_0}{p-p_0})\) is the agent’s expected utility.

Directly using the continuation values for the state variables, the problem in the previous period is:

\[
\mu(n-1, \gamma, \delta, \lambda) = \min_{\lambda \geq 0} v(p - p_0) - c_{n-1} \lambda - \delta d_{n-1} + \gamma \cdot c_{n-1} \\
+ p \max \left[ 0, v(p - p_0) - c_n \frac{p}{p-p_0} - \delta d_n - \lambda \frac{p_0}{p-p_0} d_n - \gamma \cdot c_n \frac{p_0}{p-p_0} - \lambda \frac{p-p_0}{p-p_0} d_n + \frac{p_0}{p-p_0} c_n \right] \\
+ (1-p) \max \left[ 0, v(p - p_0) - c_n \frac{p}{p-p_0} - \delta d_n - \lambda \frac{1-p_0}{1-p} d_n - \gamma \cdot c_n \frac{p_0}{p-p_0} - \lambda \frac{p-p_0}{1-p} \cdot \frac{p_0}{p-p_0} c_n \right] \\
\text{s.t.} \quad (p-p_0) \lambda \leq p (1+\gamma)
\]

We first show that the objective always decreases with \(\lambda\) and thus the wage constraint must bind:

- If both continuations in period \(n\) are negative, the effect of \(\lambda\) on the objective is \(-c_{n-1} < 0\)
- If only the continuation in period \(n\) after success is positive, the effect of \(\lambda\) on the objective is

\[-c_{n-1} - p \cdot \frac{p_0}{p-p_0} d_n + \frac{p_0}{p} c_n = -(1-p_0) c_{n-1} < 0\]

- As the value after success is higher than the value after failure (lemma 4), if the value after failure is positive the value after success is also positive. Thus, the last case to consider is that both continuations are positive. The value of the continuation after failure is decreasing in \(\lambda\) by \(\frac{1-p_0}{1-p} d_n + \frac{p_0}{1-p} c_n\) whenever it is non-zero. Thus, the overall effect of \(\lambda\) on the objective must be negative.

Plugging in \(\lambda = \frac{p}{p-p_0} (1+\gamma)\) in the problem for period \(n-1\) obtains the result in the lemma. \(Q.E.D.\)

**Proposition 6** In a two period problem, for any \(\varepsilon > 0\) consider increasing \(c_1\) by \(\varepsilon\) and decreasing \(c_2\) by \(\varepsilon\). The change strictly increases expected profits if and only if:

- The optimal contract asks the agent to work after failing in the first period; or
- The optimal contract asks the agent to work only after success in the first period and \(p + p_0 > 1\)

**Proof:** The first condition is equivalent to

\[ v(p - p_0) \geq \overline{\nu} \]

The second condition is equivalent to

\[ v(p - p_0) \in \left[ c_2 + \frac{p_0}{p-p_0} (c_2 - c_1), \overline{\nu} \right] \]

With

\[ \overline{\nu} \equiv c_2 \frac{p}{1-p} + \frac{2-p}{p-p_0} - c_1 \frac{1-p_0}{p-p_0} \frac{p}{1-p} \]
By the lemma, if \( v(p - p_0) \geq \pi \), the dual value after failure is positive and the contract asks the agent to work in the second period for any continuation. The proof for proposition 4 applies and the payment in both periods is \( \frac{c_2}{p - p_0} \). Thus, the payoff to the principal decreases in \( c_2 \) and is not affected by \( c_1 \). As \( \pi \) increases in \( c_2 \) and decreases in \( c_1 \), applying the change proposed in the proposition will not violate the condition.

If

\[
v(p - p_0) \in \left[ c_2 + \frac{p_0}{p - p_0} (c_2 - c_1), \pi \right]
\]

The dual value is positive after success and negative after failure. In this case the expected profit is

\[
\mu = v(p - p_0) - \frac{p}{p - p_0} c_1 + p \cdot \left( v(p - p_0) - c_2 - \frac{p_0}{p - p_0} (c_2 - c_1) \right)
\]

The effect of cost changes on profits is:

\[
\frac{\partial \mu}{\partial c_1} = -\frac{p}{p - p_0} + \frac{p_0}{p - p_0} = -\frac{p}{p - p_0} (1 - p_0)
\]

\[
\frac{\partial \mu}{\partial c_2} = -p \left(1 + \frac{p_0}{p - p_0} \right) = -p \frac{p}{p - p_0}
\]

Thus, the change proposed in the proposition increases expected profits iff

\[
\frac{p}{p - p_0} (1 - p_0) < \frac{p}{p - p_0} p
\]

Or simply

\[
p + p_0 > 1
\]

Q.E.D.

APPENDIX B: EXTENSIONS

B.1. Multiple Outcomes

This section shows that the optimal contract has all the same features, including sufficiency of LDIC, for any finite outcome space. In particular, let \( p_i \) be the probability of outcome \( i \) with effort and \( \hat{p}_i \) without effort.\(^{27}\) Note that \( \sum_i p_i = \sum_i \hat{p}_i = 1 \). We assume that \( p_i \neq \hat{p}_i \). The revenue return from effort is \( q \cdot \sum_i v_i (p_i - \hat{p}_i) \) which is assumed to be positive. Refer to \( \frac{p_i}{\hat{p}_i} \) as outcome \( i \)'s likelihood ratio (LR). To sidestep trivial indifferences between most desirable outcomes, assume that there is

\(^{27}\)That is \( p \), and \( p_0 \) from the two outcome model translate to \( p_1 \) and \( \hat{p}_1 \) here. The failure probabilities \( 1 - p \) and \( 1 - p_0 \) translate to \( p_0 \) and \( \hat{p}_0 \) here.
a 'most likely' outcome $i^\ast$ with a strictly higher LR than all others:

$$\forall i \neq i^\ast : \frac{p_i}{\hat{p}_i} > \frac{p_{i^\ast}}{\hat{p}_{i^\ast}}$$

The dual problem is easily derived below. All the results obtained for the binary outcome case are generalized. For convenience, these are summarized in the next proposition.

**Proposition 7** In the multiple outcome case:

1. LDIC are sufficient
2. The dynamic dual problem has the same state variables and properties (monotonicity and single crossing) as in the binary outcome model
3. The optimal contract pays the agent only after outcome $i^\ast$
4. If the agent is paid in a period, the continuation after that period is fixed up to $N(\delta)$ given below. The agent is paid $c_{N(\delta)}p_{i^\ast} - \hat{p}_{i^\ast}$ in all remaining periods in which outcome $i^\ast$ is obtained.

The exposition below shows that the proof is a straightforward extension of the two-outcome case. A detailed proof is therefore not provided. The dynamic problem is

$$V(n, \overline{\mu}, U, \overline{D}) = \max_{(U, \overline{D}, \overline{\mu}, \overline{q}, w, \overline{D}) \geq 0} q \cdot \sum_i v_i (p_i - \hat{p}_i) - \sum_i p_i \cdot w_i + \sum_i p_i V(n + 1, q, U^i, \overline{D}^i)$$

subject to

- Probability ($\mu$) $q \leq \overline{\mu}$
- FDIC $\forall s \leq n - 1$ ($\lambda_s$) $\sum_i \left(U^i + \sum_{j=0}^{s-1} D^i_j + w^i\right) \cdot (p_i - \hat{p}_i) - \hat{p}_i D^i_s - q \cdot c_{n-s} \geq 0$
- Regeneration $U$ ($\gamma$) $U = \sum_i p_i \cdot w_i - qc_n + \sum_i p_i \cdot U^i$
- Regeneration - $D$ $\forall s \leq n - 1$ ($\delta_s$) $D_s = qd_{n-s} + \sum_i p_i \cdot D^i_s$

The dual can be derived using the same methodology as in the binary outcome case. In particular, the dual has exactly the same variables $\mu, \lambda_s, \gamma$ and $\delta_s$. The laws of motion for $\gamma$ and $\delta$ derived from the FOCs for $U^i$ and $D^i_s$, are:

$$\gamma^i = \gamma - \frac{p_i - \hat{p}_i}{p_i} \sum_{s=0}^{n-1} \lambda_s$$

$$\delta^i_s = \delta_s - \frac{\hat{p}_i}{p_i} \lambda_s + \frac{p_i - \hat{p}_i}{p_i} \sum_{j=s+1}^{n-1} \lambda_j$$

28The FOCs for $U^i$ and $D^i_s$ are, respectively:

$$p_i \gamma^i + (p_i - \hat{p}_i) \sum_s \lambda_s - p_i \gamma = 0$$

$$p_i \delta^i_s - \hat{p}_i \lambda_s + (p_i - \hat{p}_i) \sum_{j=s+1}^{n-1} \lambda_j - p_i \delta_s = 0$$
The dual has $i$ wage constraints (one per possible outcome), each of the form:

$$(p_i - \hat{p}_i) \sum_{s=0}^{n-1} \lambda_s \leq p_i (1 + \gamma)$$

All constraints except for $i^*$ can be discarded. Intuitively, the agent is only paid for the outcome that has the highest LR. Formally, as $\gamma \geq -1$, the constraint can never bind for an outcome that is more likely without effort ($p_i < \hat{p}_i$). For the rest, $\frac{\hat{p}_i}{p_i} < \frac{p_i}{\hat{p}_i}$ and so, if the constraint for $i \neq i^*$ binds, the constraint for $i^*$ must be violated.

The dual objective is exactly the same as in the binary outcome case, adjusting for the additional notation and outcome space:

$$\mu \left(n, \gamma, \vec{\delta}, \vec{\lambda} \right) = \sum_i v_i (p_i - \hat{p}_i) + c_n \gamma - \sum_{s=0}^{n-1} (c_{n-s} \lambda_s + d_{n-s} \delta_s)$$

$$\quad + \sum_i p_i \mu \left(n + 1, \gamma^*, \vec{\delta}^* \right)$$

It is now clear that all the general results follow, including LDIC. In particular, only $\lambda_0 > 0$. To determine payments, observe that once the wage constraint binds, $\gamma^* = -1$ and the continuation is fixed (as in the binary outcome case). In any other outcome, the agent is not paid (the constraint does not bind), $\gamma^*>-1$ and the continuation starting in the next period is not fixed. If the contract is fixed, incentives can be provided only using same-period payments as in the binary outcome case. $N(\delta)$ is given by replacing $v(p - p_0)$ in the binary outcome case with $\sum_i v_i (p_i - \hat{p}_i)$. Standard methods now imply that the agent is only paid for the most likely outcome $i^*$ and the payment rule is derived as in the binary outcome case.

Finally, it is interesting to note that the dual state variables change between periods for any outcome (not only $i^*$). Recalling that the dual states reflect the agency costs, it is not surprising that the marginal costs of utility decreases iff the period outcome is more likely with effort than without ($p_i > \hat{p}_i$). Similarly, the information rent increases most for the outcome with the lowest LR and the effect decreases as the LR increases.

### B.2. Discounting

The standard procedure to introducing a common discount factor $\beta$ to problem 2.6 is by multiplying all the continuation terms (in the objective, the FDIC and the regeneration constraints). However, since no payments are deferred across periods, an equivalent, and much simpler way is to change the probability constraint to

$q \leq \beta \cdot q$

This completely captures the effect of discounting for both the principal and agent. With this change, $q$ is the “discounted probability” that the contract was not terminated. This has two implications for the dual. First, assuming sufficient smoothness, we now have that $\frac{d\mu'}{dq} = \beta \cdot \mu$ instead of simply $\mu$. This generates the second implication, which is that the continuation values $\mu^y$ are multiplied by the discount factor.
It is then immediate that nothing in the dual analysis is affected by discounting. The only part of the general analysis that can possibly change is the required payment identified in proposition 4, as this is based on the primal. Intuitively, discounting “dilutes” the agent’s information rent and so the agent can be paid less to forgo it. The next lemma identifies the solution.

**Lemma 12** If the principal and agent have a common discount factor \( \beta \), then after the first payment was made, the contract is fixed to \( N(\delta) \) as in proposition 4 and the payment in period \( n \) is defined by

\[
W_{N(\delta)} = \frac{c_N(\delta)}{p - p_0}
\]

\[
W_n = \beta W_{n+1} + (1 - \beta) \frac{c_n}{p - p_0}.
\]

**Proof:** As the contract after payment still does not depend on new outcomes, the relevant IC is still as in the proof of proposition 4:

\[
(p - p_0) w_h - q_h c_h \geq D^1
\]

Discounting means that \( q_{(h,y)} = \beta \cdot q_h \). Thus, at any period \( n \), for a contract that will end in period \( N(\delta) \)

\[
D^1 = q_h \sum_{m=n+1}^{N(\delta)} \beta^{(m-n)} d_m = q_h \sum_{m=n+1}^{N(\delta)} \beta^{(m-n)} (c_m - c_{m-1})
\]

So,

\[
w_h = \frac{c_h q_h + q_h \sum_{m=n+1}^{N(\delta)} \beta^{(m-n)} (c_m - c_{m-1})}{p - p_0}
\]

As \( w_h = W_h \cdot q_h \) where \( W_h \) is the actual payment for success in the history \( h \), divide both sides by \( q_h \):

\[
W_h = \frac{c_h + \sum_{m=n+1}^{N(\delta)} \beta^{m-n_h} (c_m - c_{m-1})}{p - p_0}
\]

The result for \( W_{N(\delta)} \) is immediate. Now observe that

\[
\beta W_{n+1} = \frac{\beta c_{n+1} + \sum_{m=n+2}^{N(\delta)} \beta^{m-n_h} (c_m - c_{m-1})}{p - p_0}.
\]
So
\[ W_n - \beta W_{n+1} = \frac{c_n + \sum_{m=n+1}^{N(\delta)} \beta^{m-n_h} (c_m - c_{m-1}) - \beta^m c_{n+1} - \sum_{m=n+2}^{N(\delta)} \beta^{m-n_h} (c_m - c_{m-1})}{p - p_0} \]
\[ = \frac{c_n + \beta c_{n+1} - \beta c_n - \beta c_{n+1}}{p - p_0} \]
\[ = (1 - \beta) \frac{c_n}{p - p_0} \]

Q.E.D.

APPENDIX C: IMPLEMENTATION DETAILS

This appendix provides details regarding the implementation. The dual space \((\gamma, \delta)\) was approximated by a vector of points distributed on each dimension. As usual, the solution iterated backward starting from the last possible \(n\), using value iteration. For every point, a solution required convergence to the optimal policy within \(10^{-7}\).

The coordinates for \(\delta\) were distributed uniformly between zero and \(\delta\) for which \(\mu(n, -1, \delta) = 0\). The results reported here used at least 200 \(\delta\) values.

To determine the coordinates for \(\gamma\), first define \(\overline{\gamma}\) by \(\mu(n, \overline{\gamma}, 0) = 0\). If \(\overline{\gamma} < 2\), the coordinates for \(\gamma\) were distributed uniformly (at least 200 values) between -1 and \(\overline{\gamma}\). If \(\overline{\gamma} > 2\) the coordinates for \(\gamma\) were made of two lists (each with at least 200 values). One distributed uniformly between \(-1\) and \(2\) and the other between \(2.4\) and \(\gamma\). Some extra points were added to increase smoothness at the edges. As a result, the number of points evaluated per \(n\) using 200 as the basic coordinate count per variable were either about 100,000 (if \(\overline{\gamma} < 2\)) or about 200,000 (if \(\overline{\gamma} > 2\)). The interpolation between points used a polynomial of degree 3.

After a dual solution was found, a second algorithm traced down the tree starting from the first period using the optimal dual values. The trace stopped once the dual value was zero. The algorithm identified the optimal wage, agent’s utility and principal profit given the resulting work plan. These were compared to the values given by the dual solution. The values were typically very similar. However, the second (tracing) algorithm, did not fully adjust the contract when randomization was used and this slightly reduced the profits and work reported by it in some cases\(^{29}\).

A note about accuracy. Using a quarter of the points had a negligible effect on the accuracy of the dual value reported by the algorithm. However, it had a significant implication for the accuracy of the tracing algorithm\(^{30}\). This is not surprising. Because the dual value does not require identifying the optimal policy, the dual problem is quite flat around the optimum and choosing a slightly suboptimal \(\lambda\) in a period has very limited effect. In contrast, in the primal problem choosing a slightly suboptimal continuation value \((U^y)\) may have significant implications on the continuation. This is only mentioned as a conjecture that may be worth exploring in future work.

\(^{29}\)This happened in about half of the reported cases. When randomization was required, the tracing algorithm reported profits lower by about 2%. The only outlier was the 16 period case in which the tracing algorithm reported profits lower by 8% due to unaccounted-for randomization.

\(^{30}\)The number of points was chosen after verifying that further increase did not effect the tracing algorithm as well.